Non-Euclidean Perspective

Part 4

An Introduction
to the
Perspective Illustration
of
Non-Euclidean Geometry

by
Jim Barnes, architect

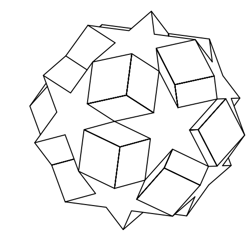


Table of Contents

Chapter 1. Getting Ready .... pages 2-8
Chapter 2. The Saccheri Quadrilateral and Image Size .... pages 9-21
Chapter 3. Appearance of Non-Euclidean Planes .... pages 22-44
Chapter 4. Straight line figures on a plane .... pages 45-65
Chapter 5. Appearance of curves with even rates of bending ... pages 66-84
Chapter 6. 3-D straight-edged forms and flat surfaces .... pages 85-135
Chapter 7. 3-D curved surfaces with even rates of bending ..... pages 136-152
Chapter 8. Other Drawing Methods .... pages 153-169
Epilogues: Why Non-Euclidean Perspective? Problems .... pages 170-171
Appendix: Mathematical Forumulae.......... pages 172-177
Bibliography .... page 178
Index .... pages 179-182

11th of February, 2017
Fourth Version of First Draft

Jim Barnes, architect (James David Barnes) Owner of Copyrights
1503 NW 39th Street; Oklahoma City, Oklahoma 73118-2603; U.S.A.
(405) 525-3069; barnes444@sbcglobal.net; http://www.7Ladders.com

Chapter 7:
The appearance of curved surfaces with even rates of bending
(Proper Spheres, Equidistant Surfaces, and Horospheres)

After one is familiar with the odd visual appearance of Non-Euclidean planes, it seems like it might be easiest to study curved figures first in 2-dimensions, with 3-dimensional versions following afterwards.
(My better order of presentation might be: Chapters 1, 6, 4, 5, then this 7.)

PROPER SPHERES:

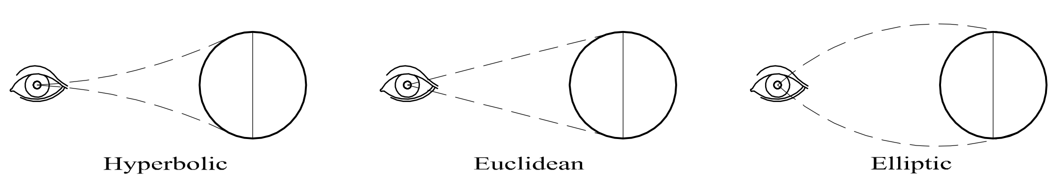
In Chapter 5 (pp. 66-84) we saw Perspective views of curved planar lines having "even rates of bending" -- Proper Circles, Equidistant Curves, and Horocyles. The rate of bending for those curves is equal at every point onlong their path. In Euclidean Geometry a curve having an even rate of bending is always a circle (or part of a circle).

The 2-Dimensional planar curves of Chapter 5 can be revolved to form 3-Dimensional curved surfaces.

DRAWING 33A (p. 68) showed Proper Circles on flat planes. Here, in *DRAWING 73,* two of the circles of *DRAWING 33A* have been revolved to form Proper Spheres. (To expose their poles to view I have arbitarily tilted their axes of rotation out of the original plane.)

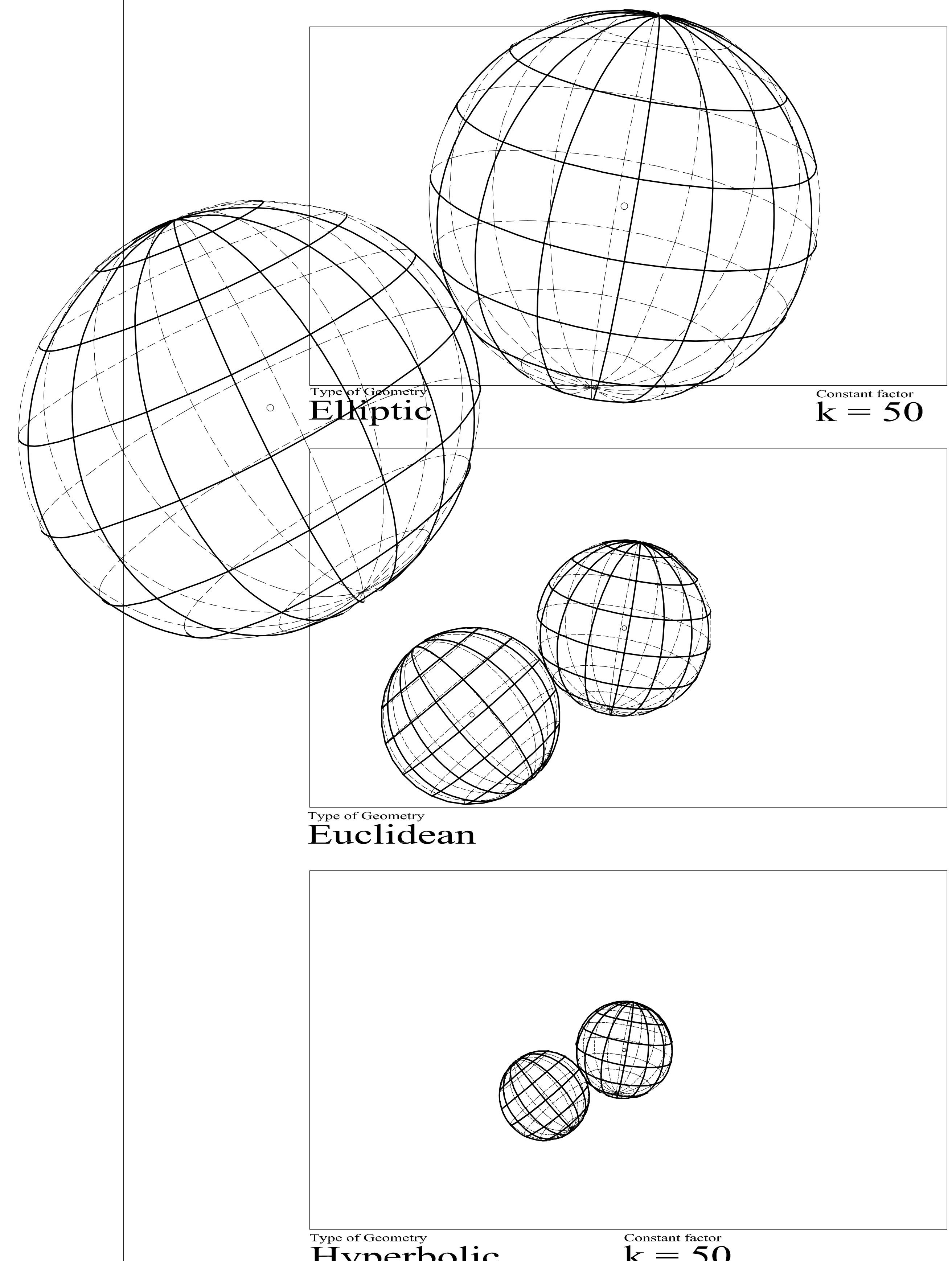
As in Euclidean geometry, in Hyperbolic and Elliptic spaces, every possible radius line is perpendicular to the surface of the sphere.

The Perspective appearance of Proper Spheres could become the subject of lengthy study. Let me quickly discuss two interesting aspects. Firstly, in Euclidean and Hyperbolic geometries it is not especially surprising that the Eye sees less than half the spherical surface (if the Proper Spheres are opaque to rays of vision); but in Elliptic Geometry the Eye often sees MORE than half the surface.



Secondly, the profile of a Proper Sphere will form a circle on the Perspective picture-plane ONLY when its center is precisely on the central line of sight (the perpendicular from the flat picture-plane through the Eye). As the sphere moves off-center its profile deforms --- in all three geometries. This unnatural distortion in Euclidean Perspective has long been considered as a defect in Perspective theory. For centuries artists have broken Perspective rules and drawn circular profiles to illustrate all spheres, regardless of their positions on the picture plane.

DRAWING 73: **PROPER SPHERES** are formed by revolving "Proper Circles" around their centers.



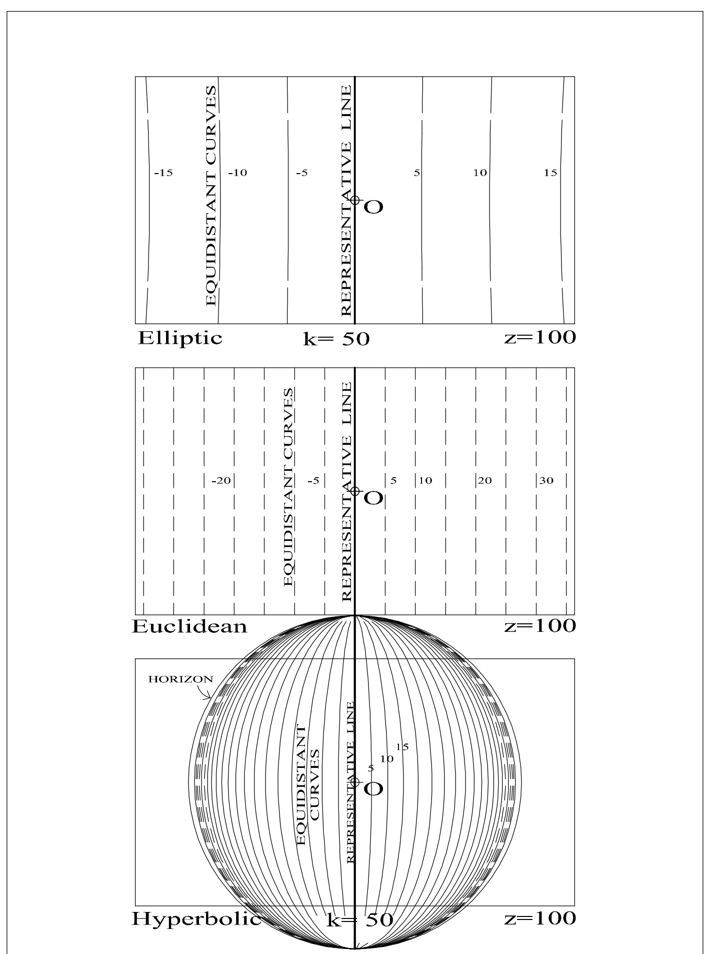
EQUIDISTANT SURFACES

1. EQUIDISTANT TUBES:

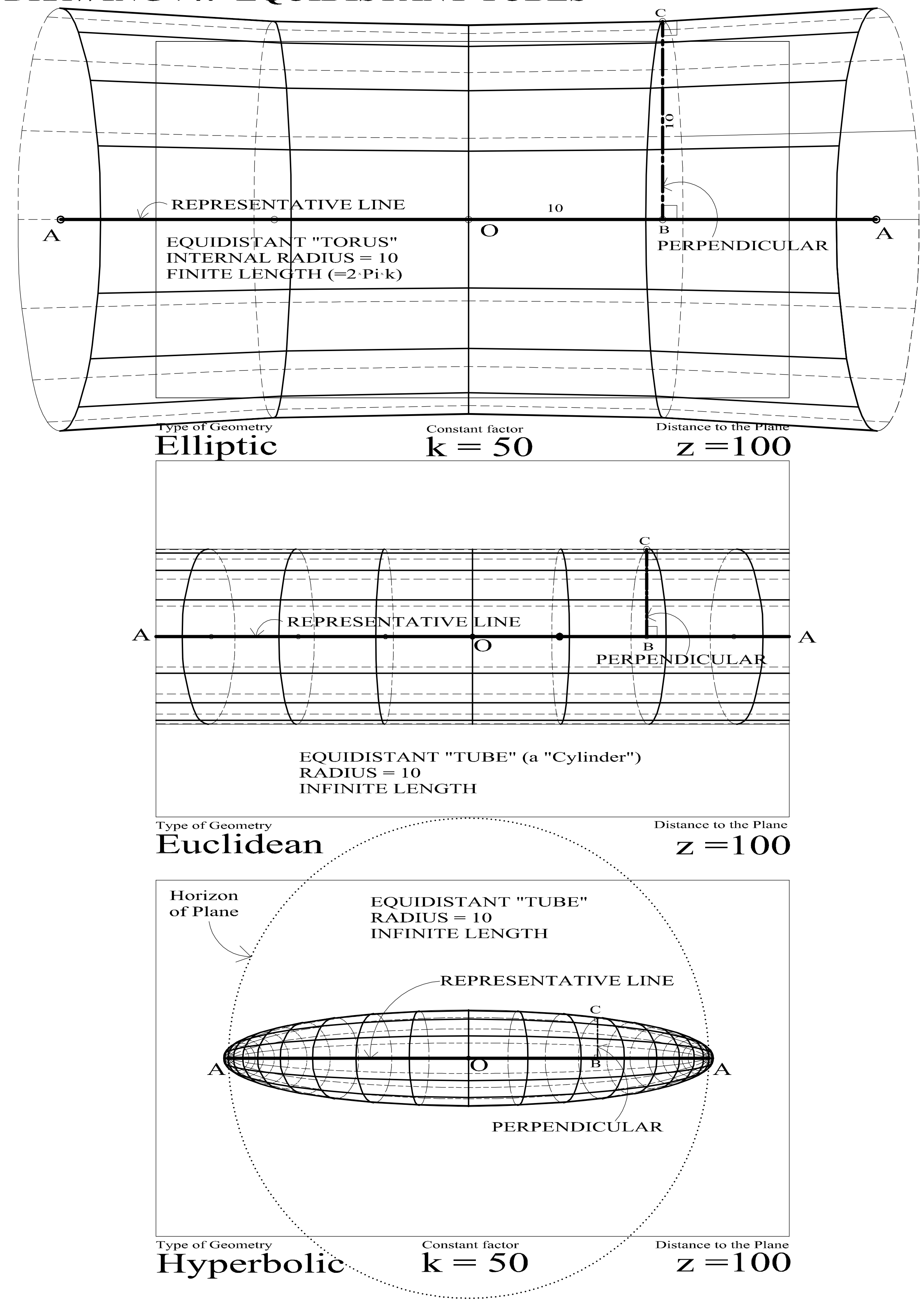
Equidistant Curves were illustrated in *DRAWINGS 33b - 36* (pp. 70-76). Those Equidistant Curves may be revolved to become three-dimensional curved sufaces in two different ways.

Here, in *DRAWING 74*, an EQUIDISTANT TUBE may be generated by revolving an Equidistant Curve along its base Representative Line. The resulting 3-dimensional surface will at every point be at an equal distance from the straight Representative Line (measured in an orthagonal manner).

DRAWING 74 generates Equidistant Tubes by taking the Equistant Curves of *DRAWING 34* (p. 72, copied in smaller size directly below) and revolving one (at distant "10") around the axis of its base straight Representative Line. (Everthing has been rotated 90 degrees to fit better into our picture frames.)



DRAWING 74: EQUIDISTANT TUBES



2. EQUIDISTANT SURFACE:

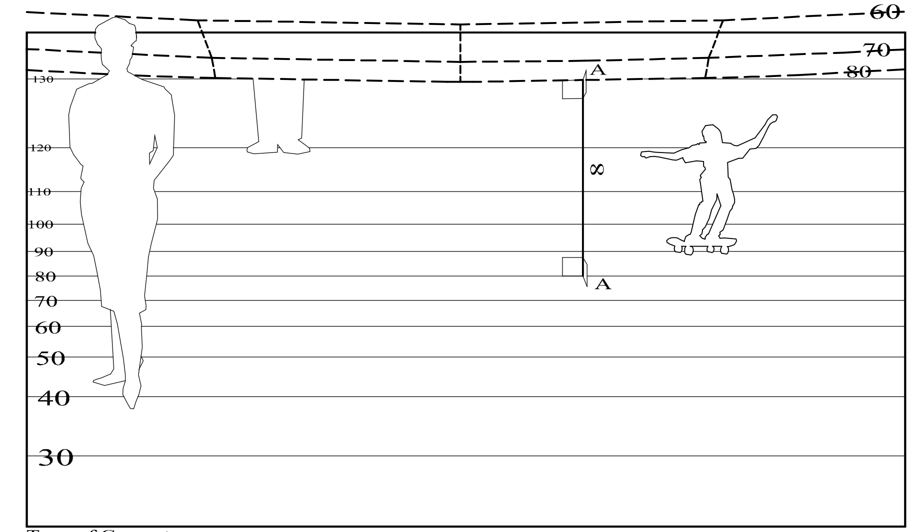
A second method of creating a 3-D Equidistant Surface is to revolve the straight Representation Line and its co-planar Equidistant Curve around a perpendicular axis. The endless straight Representation Line is spun into an endless flat plane, and the Equidistant Curve is spun into an endless curving Equidistant Surface. Every point on the new Equidistant Surface is at an equal perpendicular distance from the flat plane; and every line perpendicular to the flat plane meets the curving Equidistant Surface at a 90 degree angle.

For me, these Equidistant Surfaces are one of the best illustrations of the character of Non-Euclidean spaces. In Elliptical Geometry the flat base plane appears to encircle our Eye and its Equidistant Surface is actually a gigantic Proper Sphere. If we traveled outward, the density of the Elliptic space expands (and we appear be growing taller, though our actual height has not changed). No matter which direction we move, no matter how far we go in Elliptic space, the upward curving Equidistant Surface always remains at the same height above the flat base plane.

In Euclidean Geometry an Equidistant Surface is always a flat parallel plane.

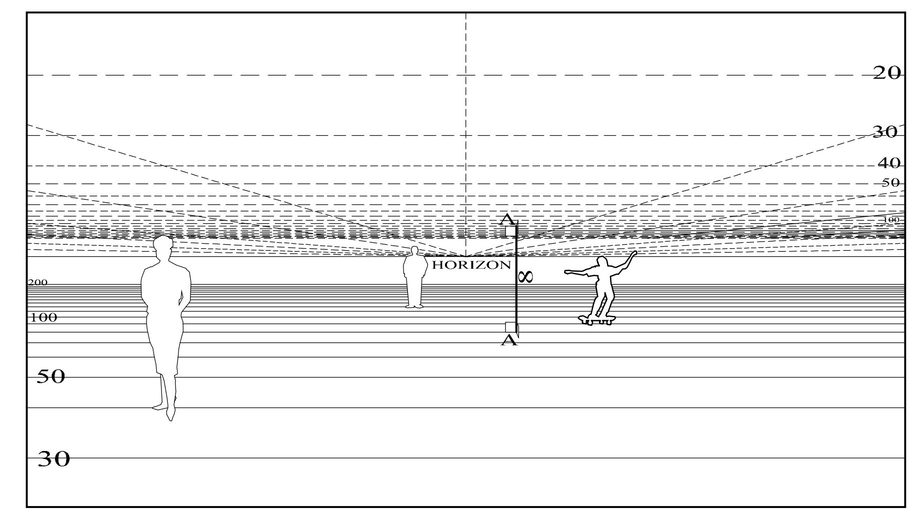
As our Perspective Eye looks out across a flat Hyperbolic plane, the infinite Equidistant Surface arches above it like an inverted saucer, ever bending downward. But if we travel out into the distance along the flat plane the Hyperbolic space gets ever denser (we appear to be growing shorter). No matter which direction we head, and no matter how far we go, in the ever-denser Hyperbolic space the length of the distance between the downward curving Equidistant Surface and the flat plane never changes. The even rate of the densification of the space is revealed in the even rate of bending of the Equidistant Surface.

***DRAWING 75:* EQUIDISTANT SURFACES are everywhere at an equal perpendicular distance from a flat base plane**

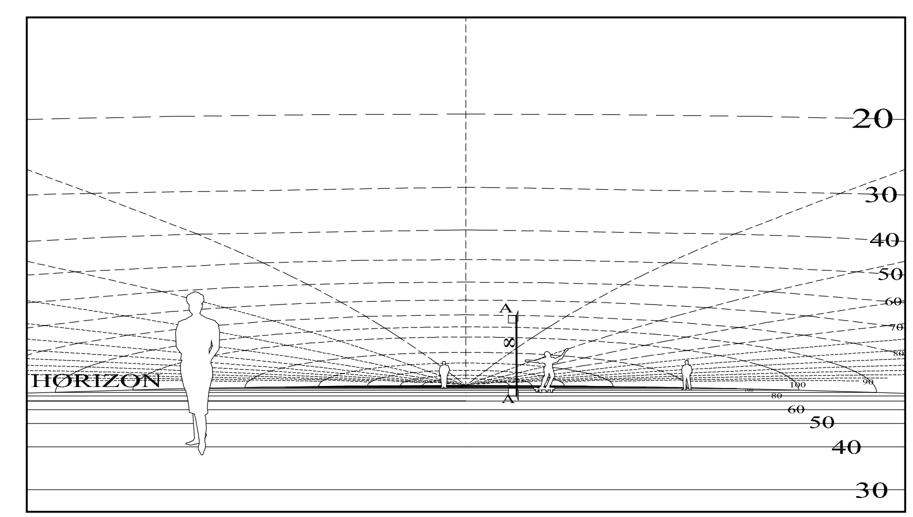


Type of Geometry
Elliptic

Constant factor
k = 50



Type of Geometry
Euclidean



Type of Geometry
Hyperbolic

Constant factor
k = 50

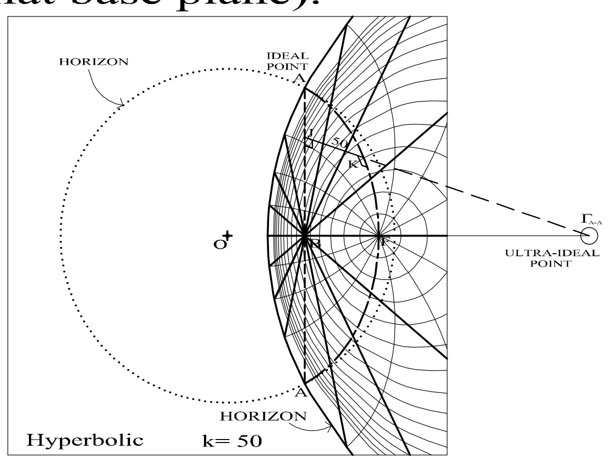
EQUIDISTANT SURFACES can always be formed in symmetrical pairs, one on each side of the flat base plane.

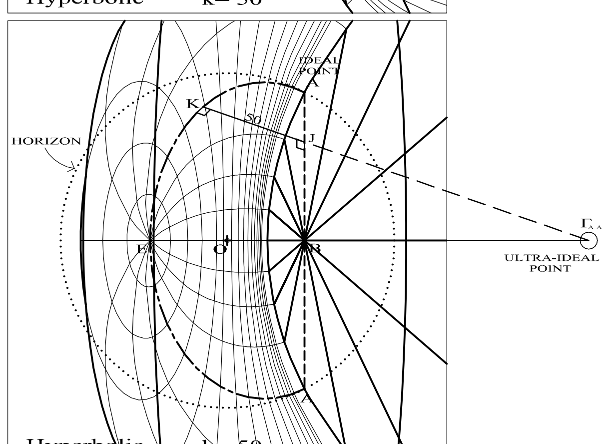
In the case of Euclidean spaces, such Equidistant Surfaces will form two flat planes parallel to the base plane.

Two endless Equidistant Surfaces in Hyperbolic space are viewed in Perspective here in *DRAWING 76*. The Representative Line (the straight base line -- "A-A") is revolved around one of its perpendiculars (line "E-F"). Point "B" is the then the center of rotation. Two equally spaced Equidistant Curves ("A-E-A" and "A-F-A" are also rotated to form two continuous Equidistant Surfaces.

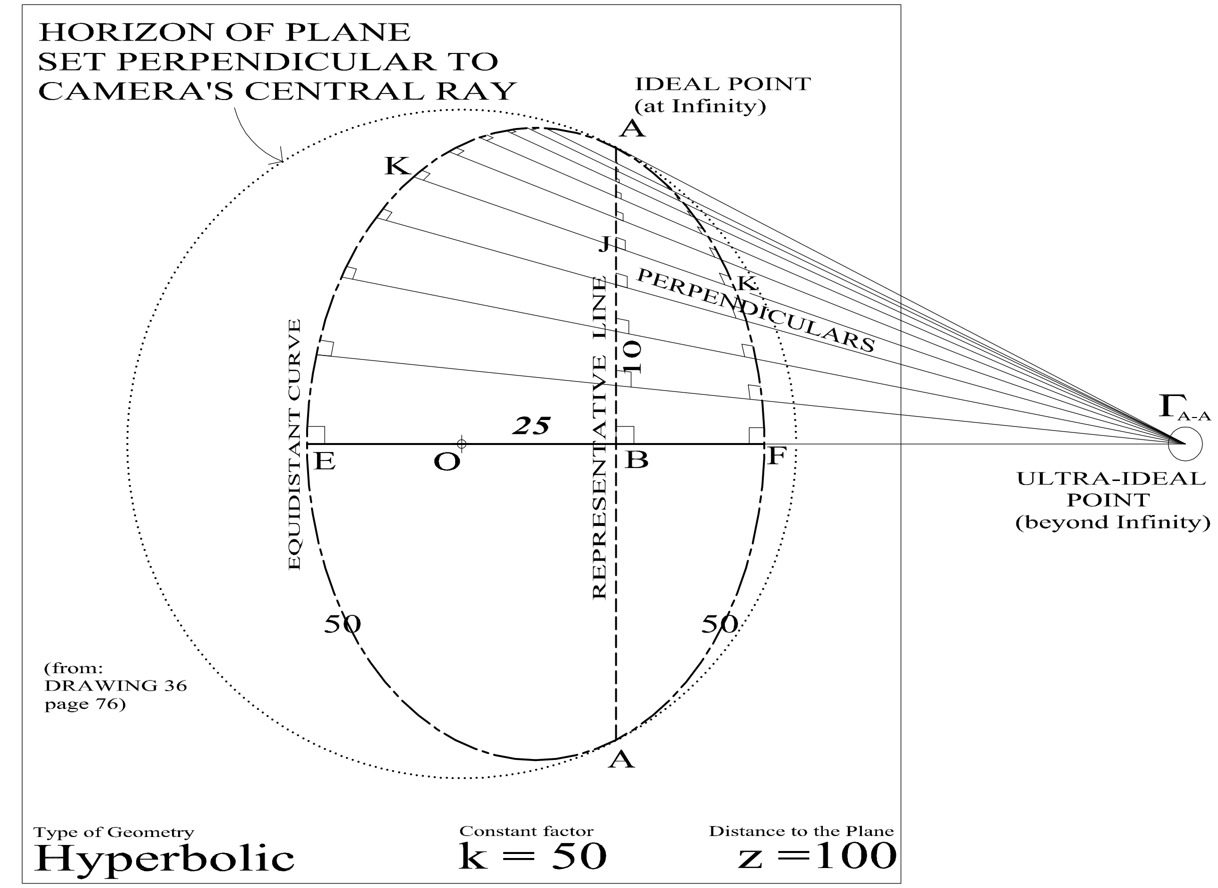
In this Perspective view the pair of Equidistant Surfaces form a single *apparent* ellipsoid (rotated ellipse) but in fact the "Ideal Points" at "A" are infinitely far away (at the Horizon) and the two Equidistant Surfaces never meet (in Proper Space).

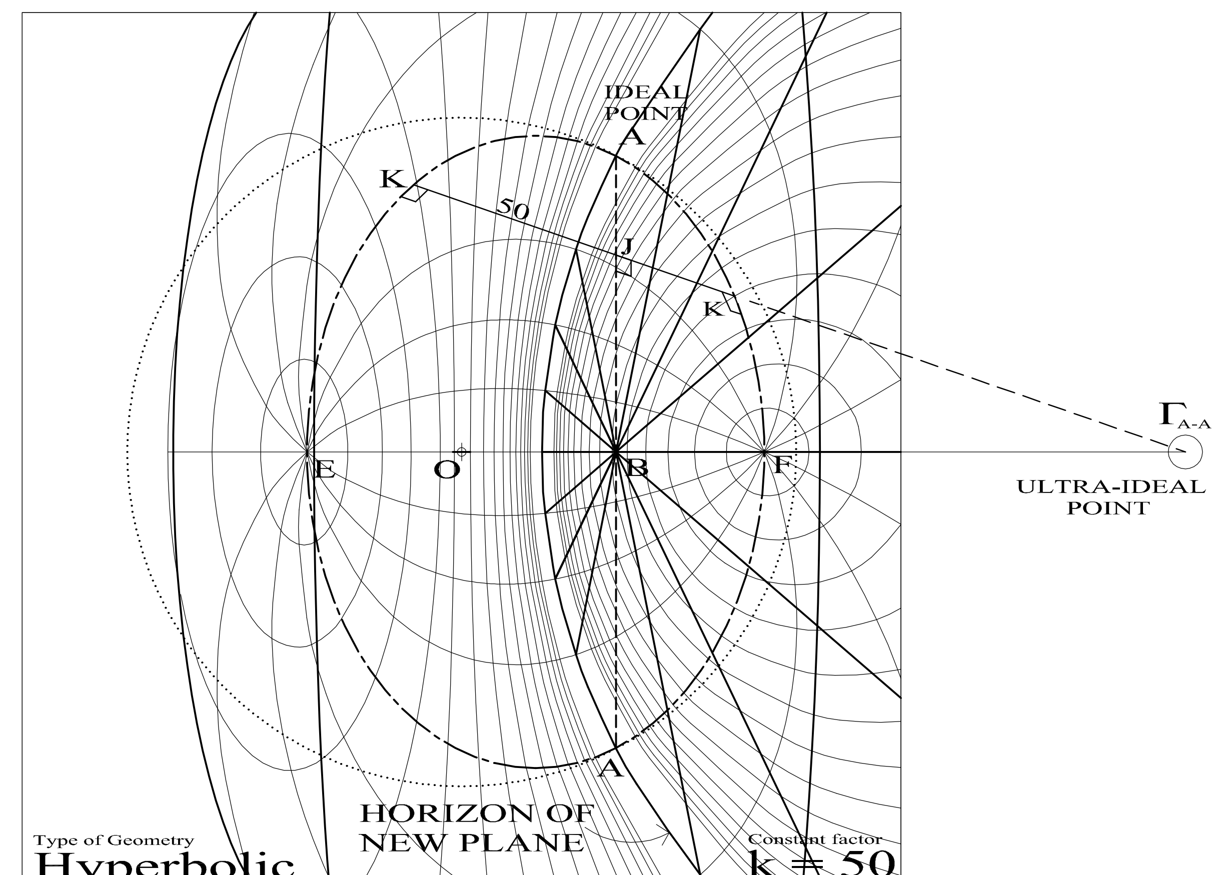
The two small drawings below separately illustrate the flat base plane and each of its two symetrical Equidistant Surfaces (one to the right, outside the base plane, and one to the left, inside the flat base plane).





DRAWING 76: EQUIDISTANT SURFACES come in pairs





EQUIDISTANT SURFACES in Elliptical Geometry:

In two-poled Elliptical Geometry a Proper Sphere with a radius of half the distance between the poles (1/4 the finite length of a line) forms a flat plane. Equidistant Surfaces on each side of the flat plane are also Proper Spheres. All three Proper Spheres have the same centers, but each surface has two centers -- one on each side. In the istropic versions of Elliptic Geometry discussed in this book all straight lines of the space have the same finite length.

I was having trouble imagining the Perspective appearance of a symmetrical pair of Equistant Surfaces on each side of a flat Elliptic plane, so I drew the series of Distance Diagrams seen here in *Drawing 77*. Using the Eye as a fixed origin point, these diagrams take a planar slice through the space and plot the distance of various points along various angles seen from the Eye. We can imagine the Eye rotating and looking any direction. When the Eye changes position we must draw a new diagram. The series shows what the Eyes sees from various positions as it travels along a perpendicular path through the flat plane.

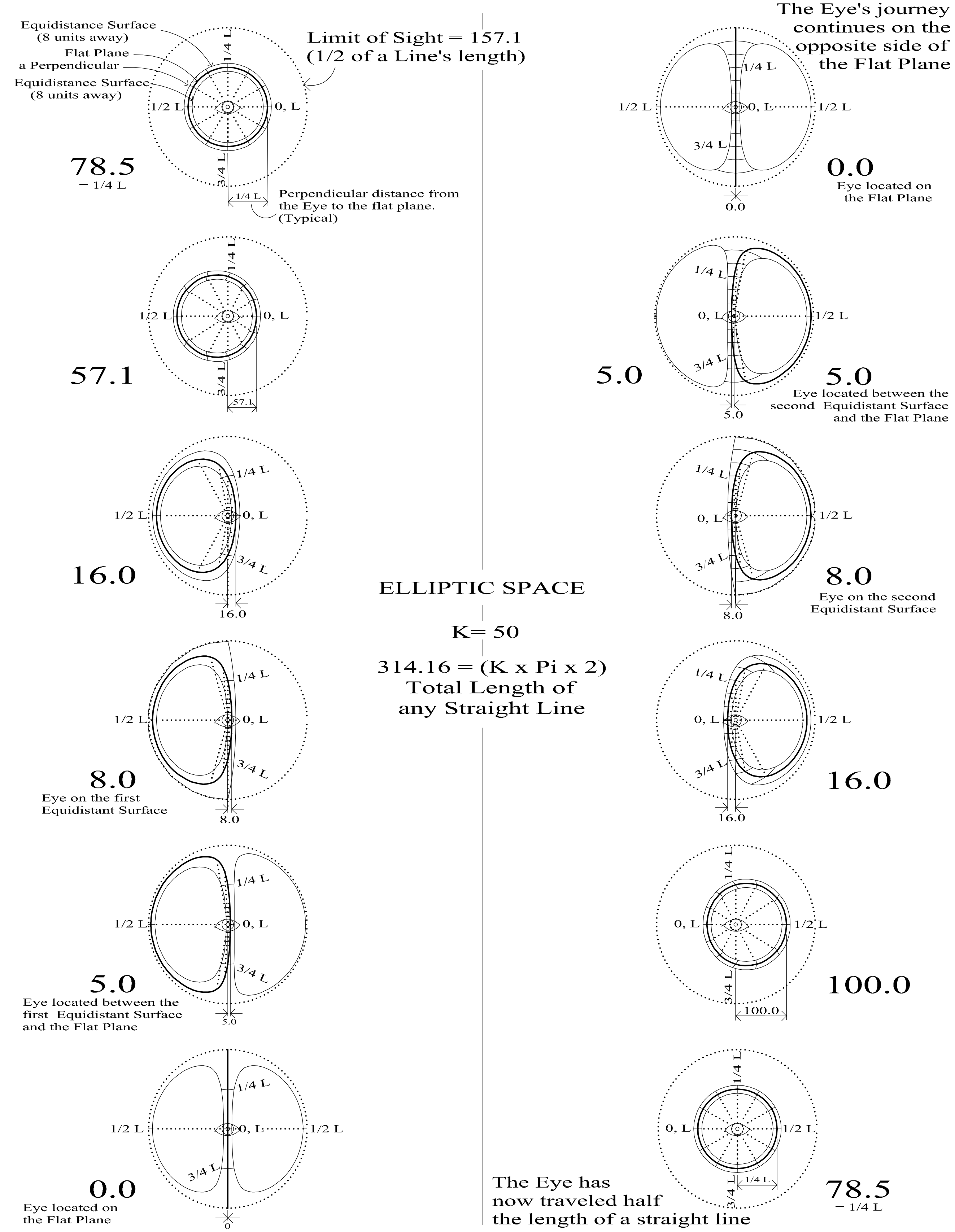
The Eye starts at 1/4 the finite length of a line away from the flat plane ("78.5", upper left). The flat plane and its Equidistant Surfaces will appear as precise spheres encircling the Eye. Equal distances measured along the surfaces will appear to the Eye within equal angles.

As the Eyes moves relatively closer to the flat plane, the encircling images swell and stretch like elastic bubbles. When the Eye is precisely upon the flat plane ("0.0"), the flat plane appears as a flat sheet extending out on either side, with barely more than a single point connecting the outer edges, stretched out across the rest of the surrounding field of view.

At the end of traveling 1/2 the length of a line ("78.5", lower right) the Eye again sees the flat plane and Equidistant Surfaces as precise spheres, but now everything is inside-out, and the Eye would need to travel the remaining 1/2 line's distance (not illustrated here), passing back through the flat plane, to return to the same view (at the same position) where it started.

This diagram shows an Eye's view out to a distance of 1/2 a line length. What does the Eye see farther out?

***DRAWING 77:* Diagrams of a plane cut through an Elliptic space, showing the distances and angles from the Eye to a flat plane and a symetrical pair of Equidistant Surfaces**



Drawing 78 takes three of the Distance-Diagrams from *Drawing 77* and doubles the length of their views. On the left are three diagrams with the Eye's sight limited to 1/2 the finite length of a line; and on the right are the same diagrams with views extended twice as far -- to the full length of a line in the Elliptic space. The outer edge of the circle-limit is then the point occupied by the Eye itself. The Eye could rotate from it's fixed position to look in any direction, and at the distance of the full length of a line of sight it would see its own back (whatever we imagine the back of an Eye would be if nothing opaque blocked the rays of vision returning to the point of their origin).

The visual location (angle and distance) of the flat plane and the symmetrical pair of Equidistant Surfaces are shown.

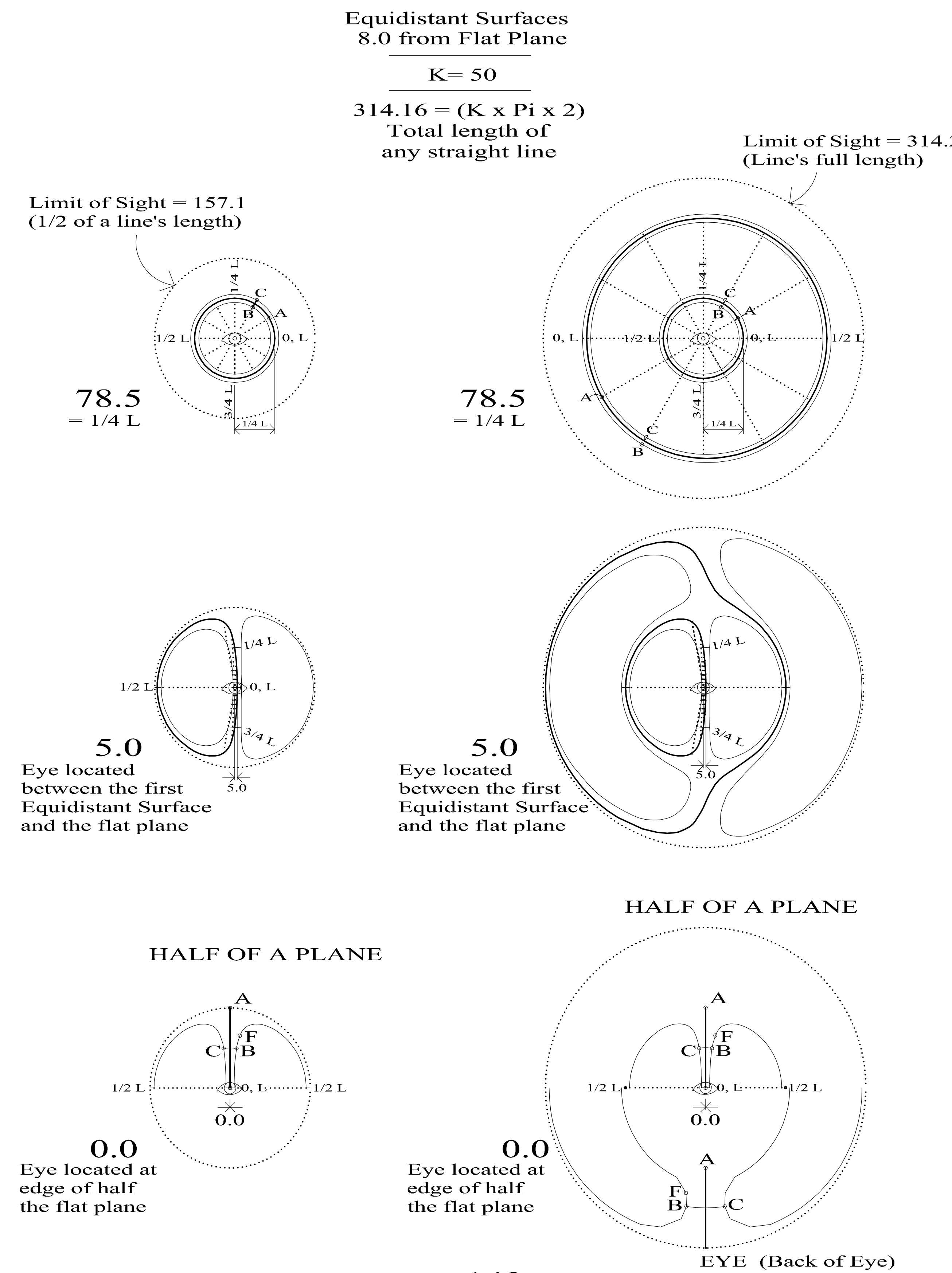
Looking farther out into space, (if everything is translucent) the Eye sees all three surfaces twice. Farther in the distance it now sees the inverted backsides of those three spheres, in an exactly opposite direction.

In the third comparative example as the Eye is positioned exactly in the flat plane, only half a line (half the flat plane) and half the corresponding Equidistant Curves are illustrated. The Eye can see to the end of half the length of the Line, or turn around and see that half the flat plane receeding back to the Eye. Past a distance of half the finite length of a line in Elliptic space, a Perspective view shows an upside-down mirrored view.

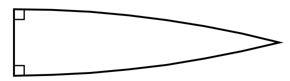
These are only Diagrams. What would complete Perspective views of these extended distances look like?

And if there were nothing blocking it, what would an Eye see if it could look still farther out into Elliptic space (rays of vision circling around more than once) ?

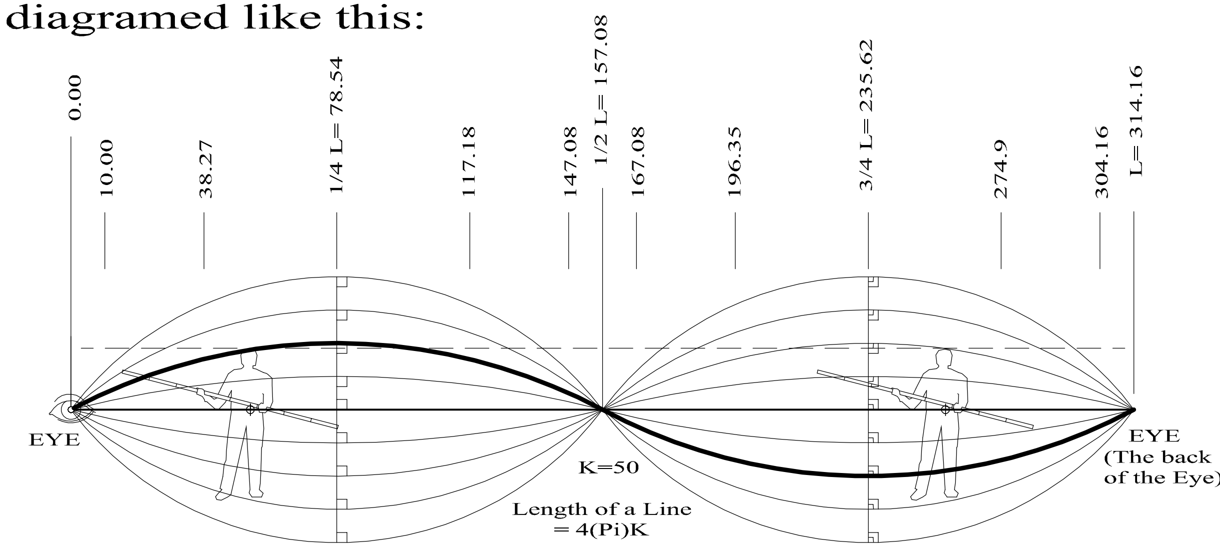
***DRAWING 78:* Three examples from DRAWING 76 with views extended out twice as far.**



Elliptic Geometry started from an initial assumption that distance will grow smaller between two co-planar straight lines set mutually perpendicular to a base line. Though the lines are straight (and will appear straight in Perpective views) we can picture that assumption of convergence in a diagram, thus:



If we position our Perspective Eye at the point where all such mutually perpendicular lines meet (called a "pole"), and if we assume that our Elliptic space has two such poles, then we have an optical condition for that Eye which may be diagramed like this:



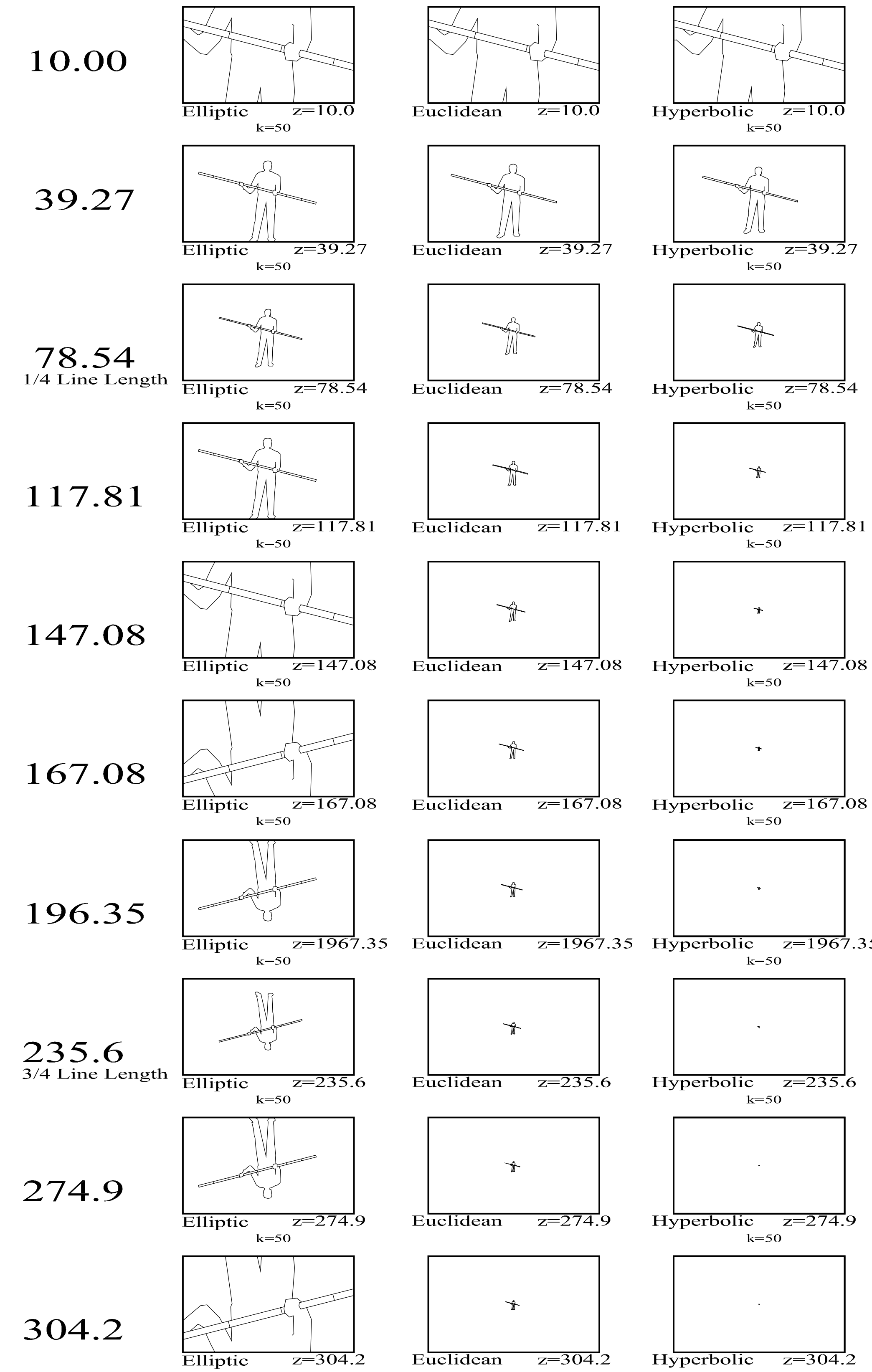
As the figure moves away from the Eye its image at first gets smaller. Then, past 1/4 line length, the image grows larger; until, at the the opposite pole (1/2 line length), the image becomes infinitely large (like pressing the camera directly against the figure). Beyond the pole opposite the Eye the figure's image appears inverted-- mirrored right-to-left and flipped top-to-bottom. The image again grows smaller as the figure continues to move farther, but again starts to grow larger past 3/4 line length.

If visual rays had finite speed and the Eye moved fast enough to permit rays to repeatedly circle through space, would the Eye see multiple images?

A Perspective illustrator may arbitrarily set a limit to the maximum length of visual rays. (This books typically uses: "1/2 line" length, in Elliptic spaces.)

This is a basic characteristic of Perpsective illustration for any Elliptic Geometry. It should have been presented early in this book, not near the end. Such is discovery.

***DRAWING 79:* Comparision of Perspective images of a figure set at varying distance from the Eye (in all 3 Geometries).**



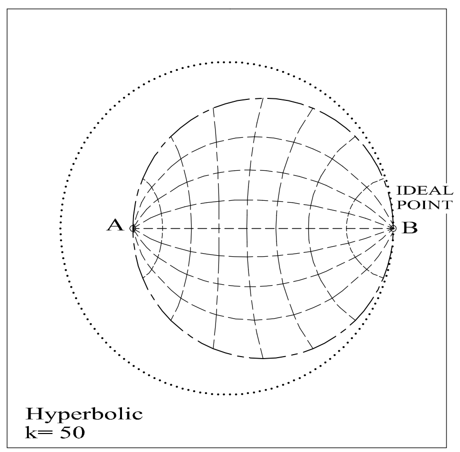
HOROSPHERE:

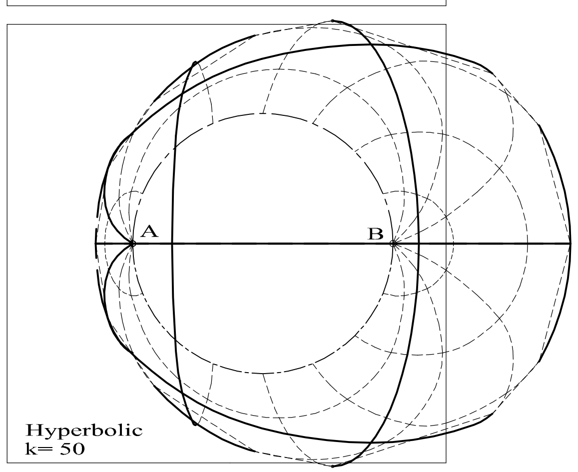
In Hyperbolic Geometery a HOROSPHERE is a Proper Sphere with radius of infinite length.

A 3-Dimensional HOROSPHERE can be formed by revolving a 2-Dimensional Horocycle around any of its radii.

Here, in *DRAWING 80*, we take the planar Horocyle drawn through two arbitrarily given points, "D" and an Ideal Point "B". For convenience we rotate the Horocyle along axis "A-B", one of its *parallel* radii. The Horosphere appears below, rendered as a gridwork of latitudes and longitudes.

On this page that same Horosphere is shown in two parts. The upper figure shows the half of the Horosphere beyond the original plane, and the lower figure shows the half in front.

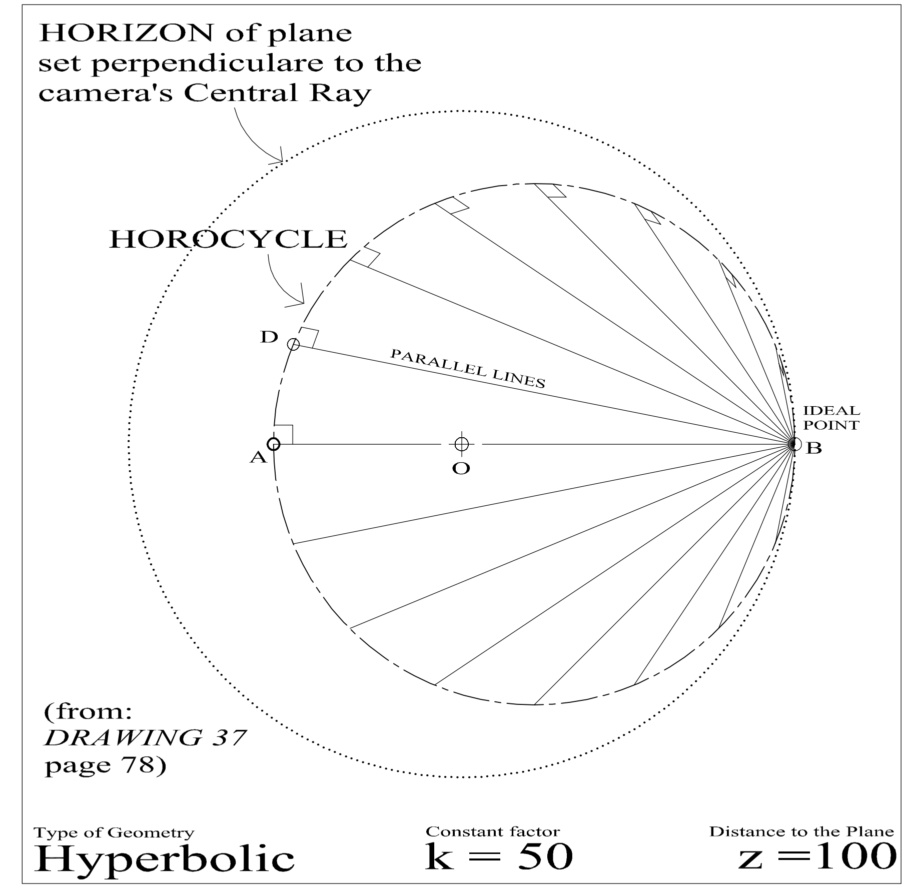


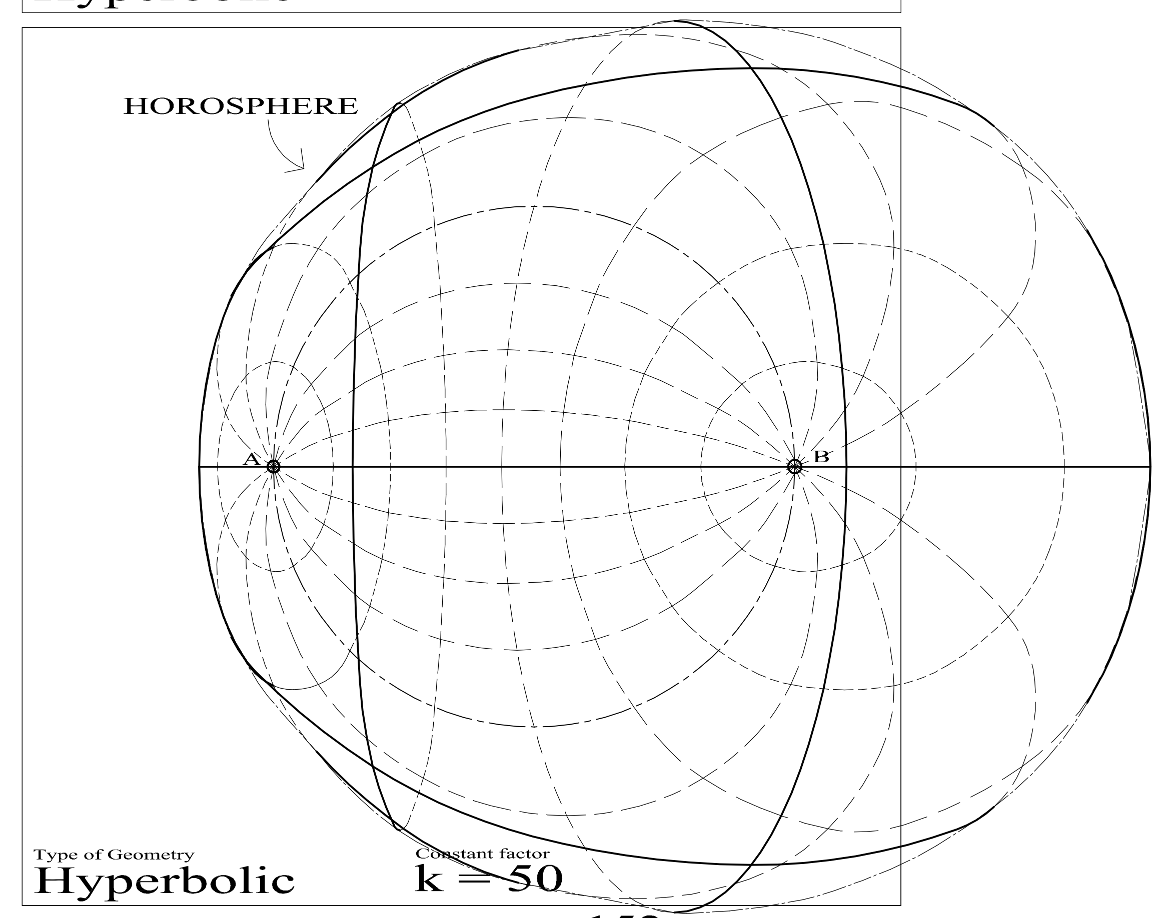


While at first it might appear that the Perpsective image has fairly evenly spaced out all the elements of the Horosphere, in fact almost its entire surface resides at point "B". With both the Horocyle and Horosphere the elements we see beyond "B" are less than one degree arc, only a fracture of their totalities.

Horospheres and their *Parallel* radii are perhaps the strangest creatures in Hyperbolic Geometry; but with their even rate of curvature being a fixed constant everwhere in the space, they turn out to be useful friends.

***DRAWING 80:* Perspective view of a HOROSPHERE**





Chapter 8:
Other
Drawing Methods

**(Distance Diagrams, Spherical Perspectives,
Glide Projections, Orthagonal Projections, and more)**

Geometry texts have always been accompanied by illustrations, and such figures merit thoughtful consideration.

This final chapter will list a few of the endless possibilities for constructing geometric visualizations, beyond the Perspective illustration system which has been the main subject of this book.

1. DISTANCE DIAGRAMS:

The simpliest way to construct precisely measured illustrations of Non-Euclidean figures is to take values of angles and/or lengths from the Non-Euclidean space and use them to plot a **DISTANCE DIAGRAM** in Euclidean space. This can be done, provided that one starts at one (and only one) point, and then graphs every other position in by consistent procedure.

Three different procedures for constructing such Distance Diagrams are shown here in *DRAWING 81*. My favorite is #3, which uses polar coordinates. All the Distance Diagrams seen elsewhere in this book use this angle-distance method.

If you imagine your human eye located precisely at the starting point, you can swivel it around and imagine the realistic retinal images of the Non-Euclidean space which it might see. But if you move your imaginary human eye to anywhere else (like viewing the Distance Diagram from above) the image will appear inappropriate -- greatly distorted.

Two-dimensional Distance Diagrams are shown in Drawing 81, but methods can be used to create three-dimensional scale models of Non-Euclidean figures as well.

Three-dimensional Distance Diagrams can be illustrated in Euclidean Perspective illustrations (as seen in *Drawings 69*, p. 132; and *Drawing 71*, p. 134.)

Alternatively, Distance Diagrams might re-arrange perpendicular axis lengths (by methods #1, or #2) into traditional *Axiometric* or *Oblique* 3-dimensional illustrations.

Additions variations to the consistent methods for creating Distance Diagrams are possible.

Construction details for the 3 Distance Diagram methods of *Drawing 81*:

1. Perpendiculars: Starting from ⊠ (point "O"), the Non-Euclidean values of "AB" and "BC" are used as Euclidean values for the corresponding Euclidean co-ordinate system, "X" and "Y", and scaled to suitable proportion to draw as a Distance Diagram.

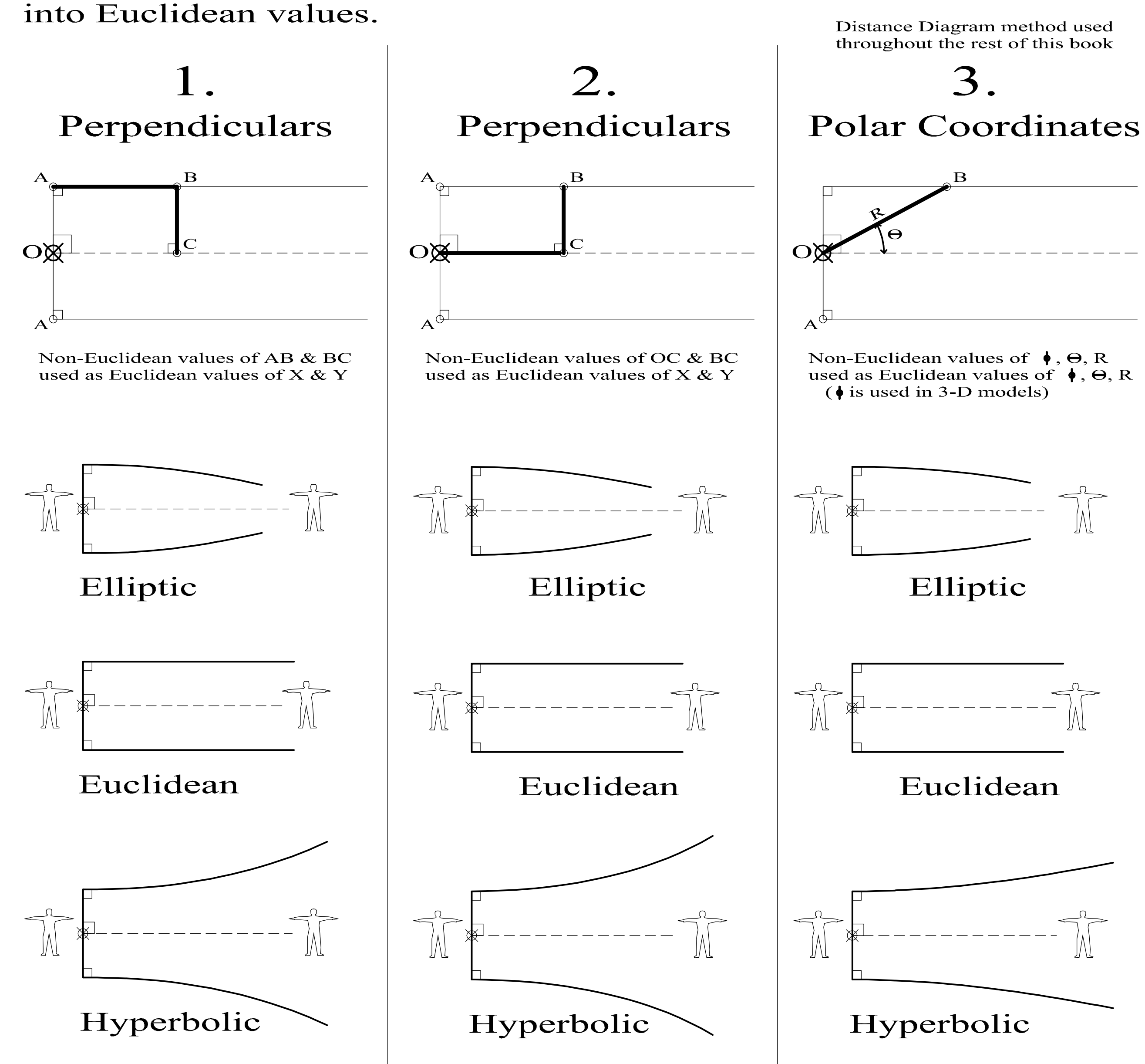
2. Perpendiculars: Starting from ⊠ (point "O"), the Non-Euclidean values of "OC" and "BC" are used as Euclidean values for the corresponding Euclidean co-ordinate system, "X" and "Y", and scaled to suitable proportion to draw as a Distance Diagram.

and the best method:

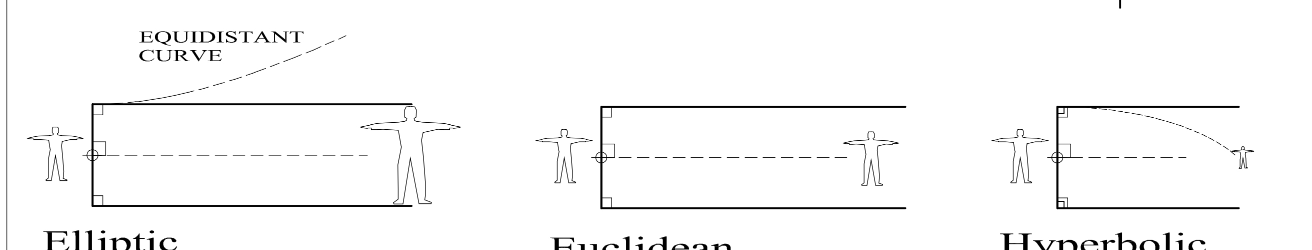
3. Polar Coordinates: Starting from ⊠ (point "O"), the Non-Euclidean polar coordinate system values (ϕ, Θ, R) describing the position of any given point "B" are used as Euclidean polar coordinate system values (ϕ, Θ, R) and scaled to a suitable proportion to draw a Distance Diagram.

***DRAWING 81:* 3 ways of drawing a DISTANCE DIAGRAM**

Using point O (at mark ⊠), as an initial fixed reference origin, any Point "B" is plotted by transposing the Non-Euclidean lengths into Euclidean values.



Comparative PERSPECTIVE views (where the Central Line of Sight is perpendicular to the plane of the objects at mark ⊕)



The Perspective illustrations in this book all use the same set-up, so that the reader may compare pictures as if they had been photographed by one camera. Some pictures have been reduced (or enlarge) and in some pictures the rectangular format has been expanded into a square -- otherwise the format producing all the Drawings of the first seven chapters of this book is uniform.

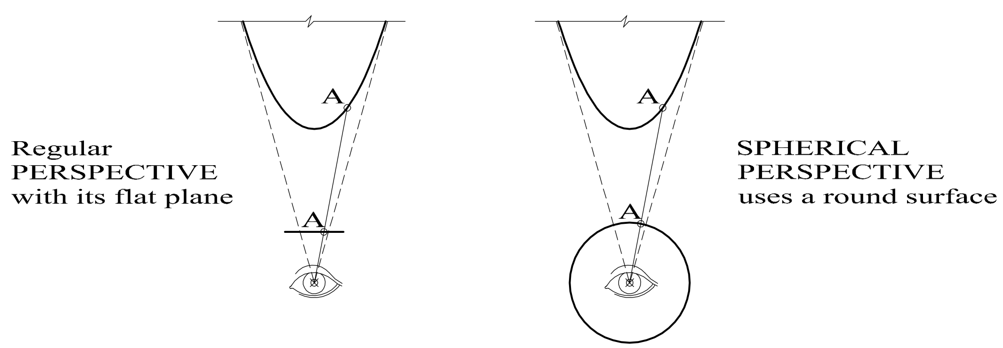
Perspective illustrations, however, may be set-up in other ways. The distance from the aperature (Eye) to the picture plane may be varied, the picture plane may be tilted, shifted aside, or assigned various shapes. Much of the art of Perspective illustration is in deciding how to set up the Perpsective geometry. The standard rules of Perspective require that the picture plane must remain flat and the cone of vision should not exceed approximately 60 degrees overall width, otherwise there are innumeral possibilities.

---

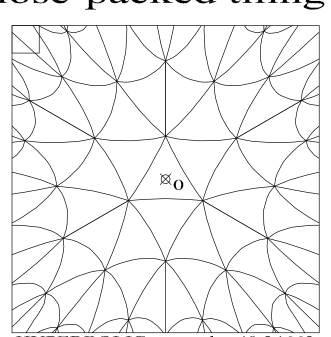
2. SPHERICAL PERSPECTIVES:

(also called "Curvilinear Perspective" or "Wide Angle" pictures)

In Spherical Perspective the flat picture plane of our regular Perspective is generalized into a sphere, a clear globe surrounding the Eye through which rays of vision pass at specific points.



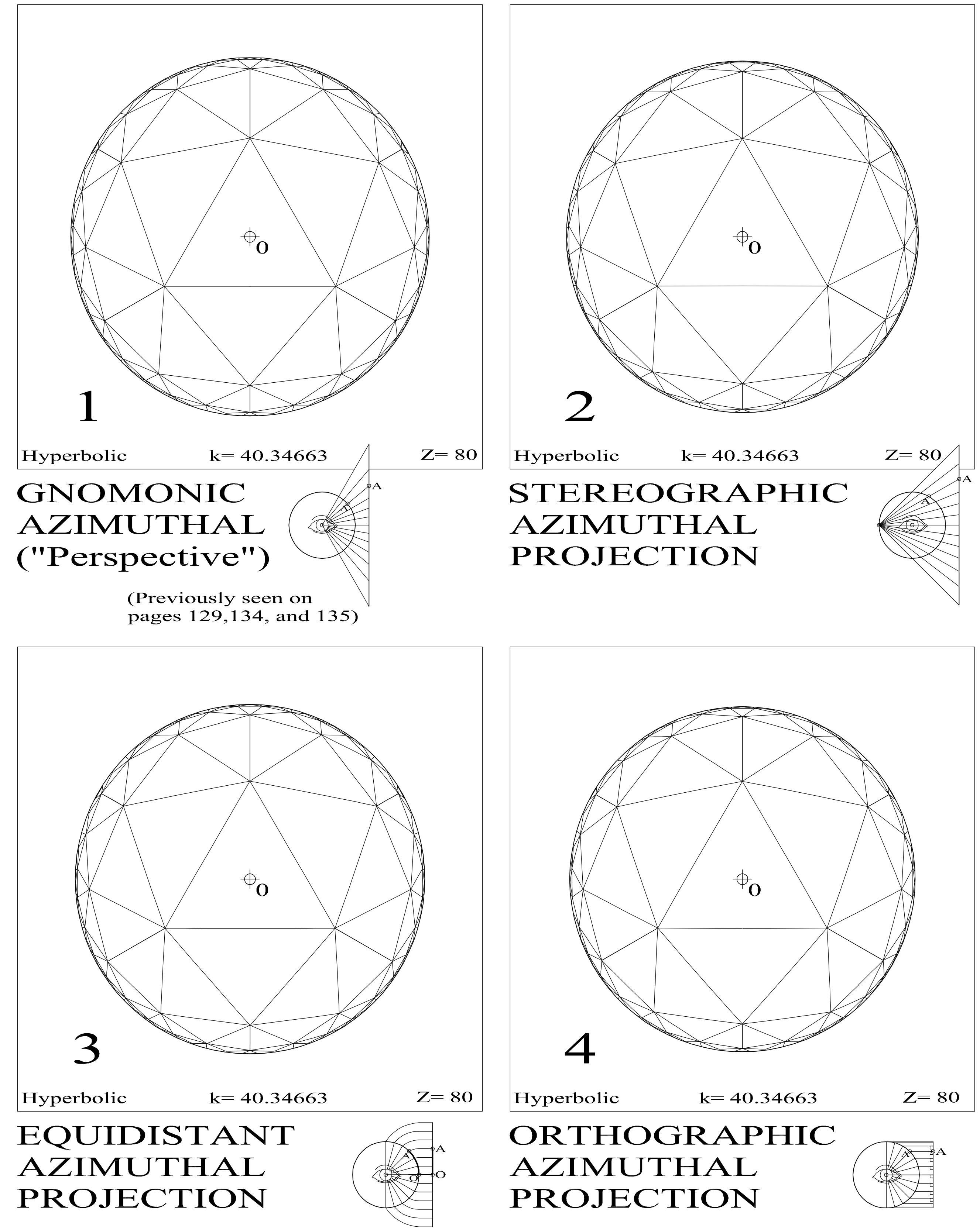
Flattening the sphere into a 2-dimensional surface is exactly the same problem as flattening the round surface of the Earth into a map. The systems of geometrical cartography are endless in number. *Drawing 82* shows four typical examples of "azimuthal" Spherical Perspectives. Adjusted to equal sizes, they each illustrate a Hyperbolic plane with a close-packed tiling of equilateral triangles.



DISTANCE DIAGRAM
(angle-distance method)
-- Equilateral Triangles
on a Hyperbolic Plane

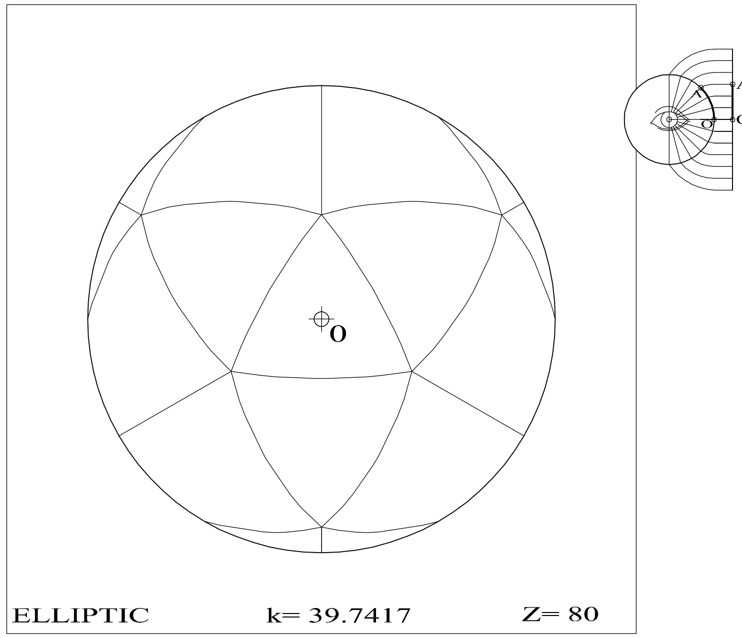
Our Perspective illustration method turns out to be one version of Spherical Perspective, the "Azimuthal Gnomonic Projection" method (shown in *Drawing 82* as example "1".)

DRAWING 82: **4 Different SPHERICAL PERSPECTIVE views of the tiling of Equilateral Triangles in Hyperbolic space (by means of 4 different Azimuthal projection methods)**

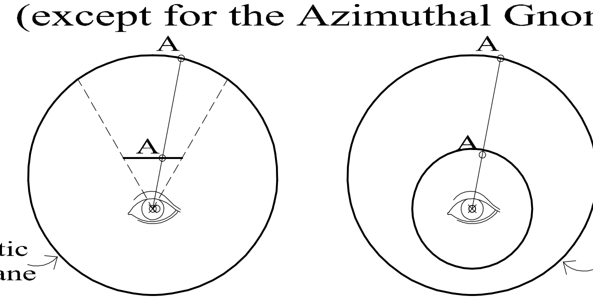


DRAWING 83: **SPHERICAL PERSPECTIVE view of one-half of an Elliptic plane tiled with equilateral triangles, by means of an Azimuthal Equidistant Projection.**

(Previously seen illustrated in Perspective on page 133.)

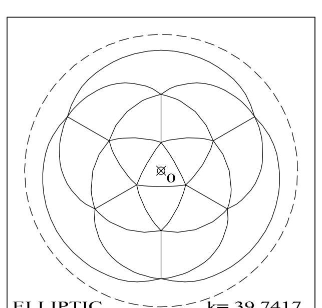


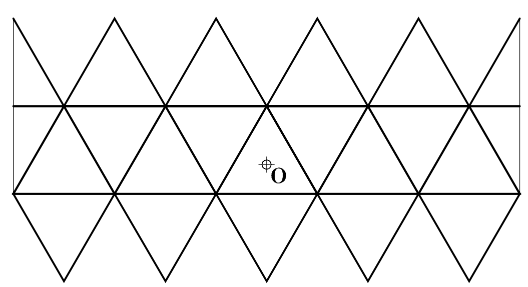
Perspective views of Elliptic space have trouble seeing sufficient portions of a flat plane encircling the Eye. Spherical Perspectives offer methods to illustrate wide-angle views, though they sacrifice the ability to illustrate straight lines in the object as straight lines on the picture plane (except for the Azimuthal Gnomonic version -- Perspective).



Here are five examples of Spherical Perspective views of an Elliptic space, including three that illustrate the entire space. 3-dimensional spaces can also be illustrated with all such Spherical Perspective methods.

DISTANCE DIAGRAM (angle-distance method, from point "O" on the plane) -- Equilateral Triangles on an Elliptic plane

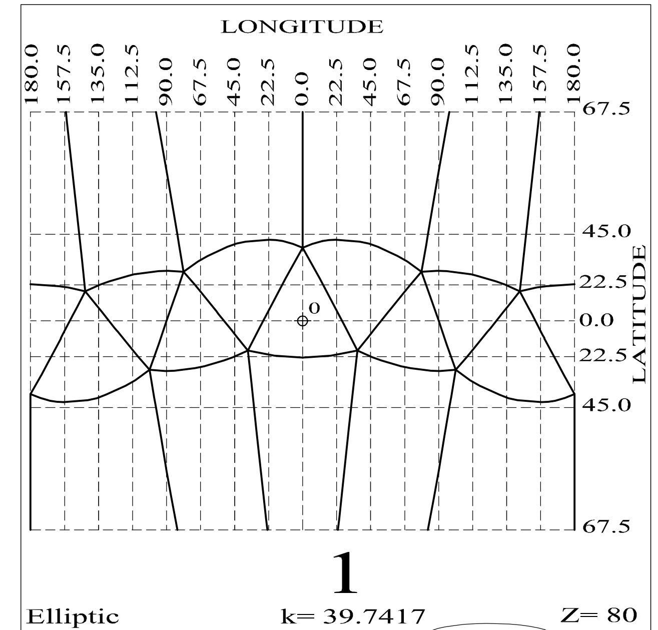




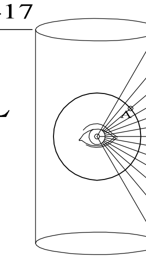
MAP OF ICOSAHEDRON

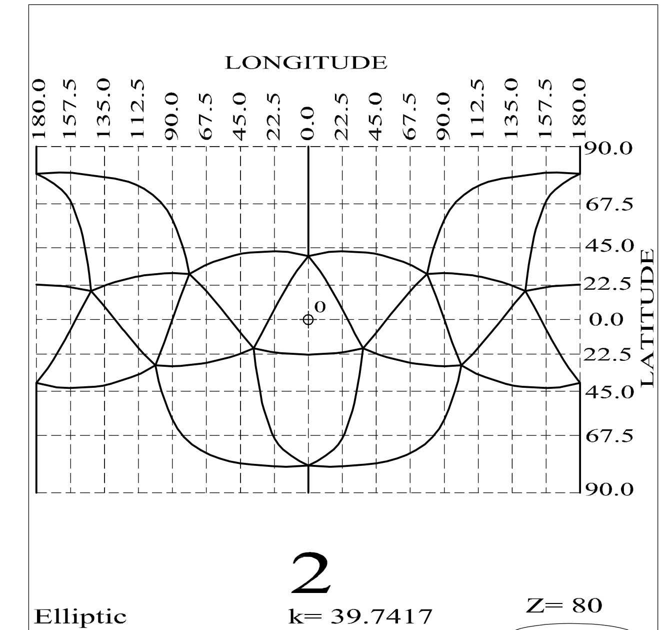
Doing these drawings, I was surprised that the 'tiling' of equilaterial triangles in Elliptic space was an Icosahedron. All the close-packed tilings of equilateral polygons must similarly be Platonic solids. Proper Sphere, Platonic solid, and flat plane -- all at the same time.

DRAWING 84: **Four SPHERICAL PERSPECTIVE views of the tiling of equilateral triangles on a flat Elliptic plane, by means of four different cylindrical projection methods.**

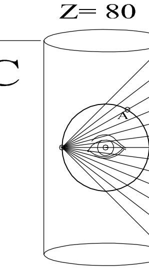


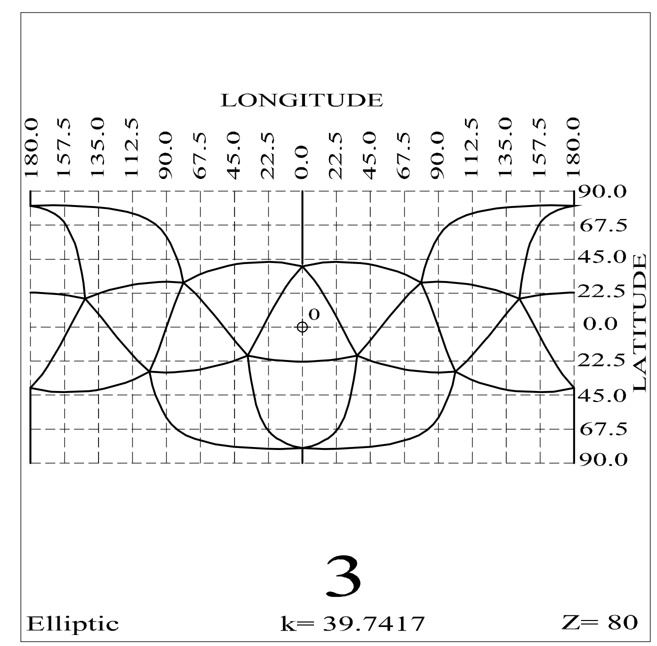
GNOMONIC CYLINDRICAL PROJECTION



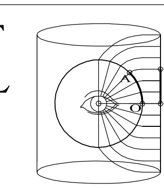


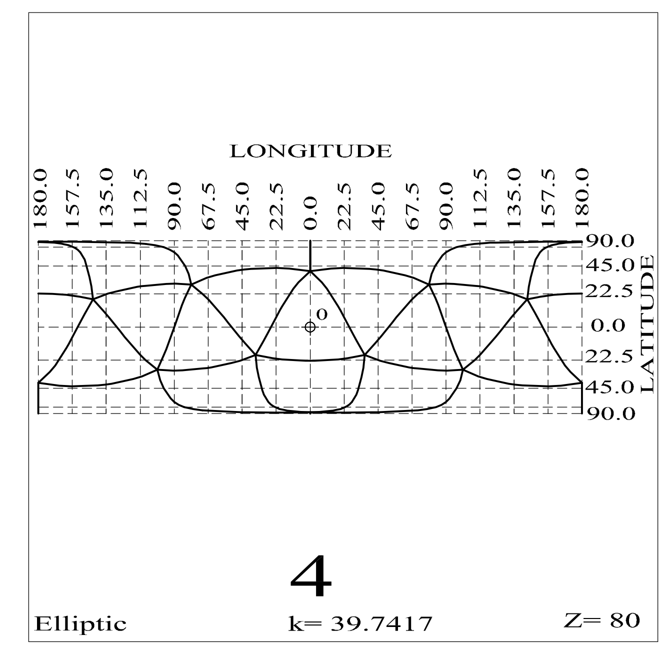
STEREOGRAPHIC CYLINDRICAL PROJECTION



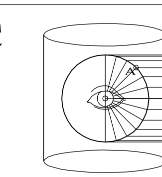


EQUIDISTANT CYLINDRICAL PROJECTION



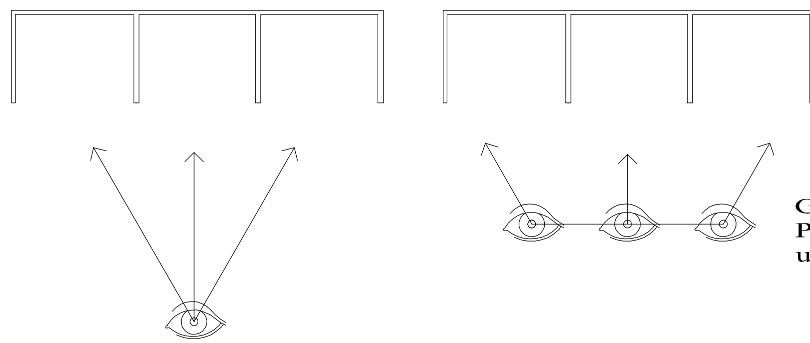


ORTHOGRAPHIC CYLINDRICAL PROJECTION



3. GLIDE PROJECTIONS:

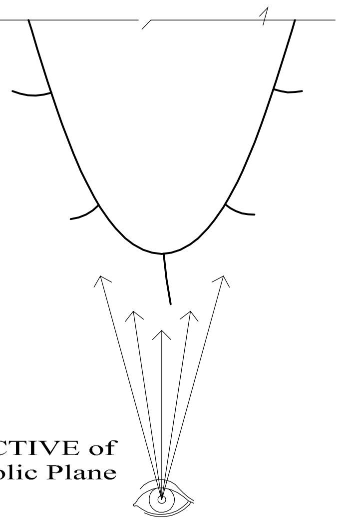
Instead of one Eye set at a single fixed position, GLIDE PROJECTIONS are a family of drawings where the Eye glides along a path in front of the object. (Reference: *Glide Projection: Lateral Architectural Drawing*, by Kevin Forseth, 1984.) In its simpliest versions an illuminating slit sweeps a surprisingly realistic view across the drawing surface.



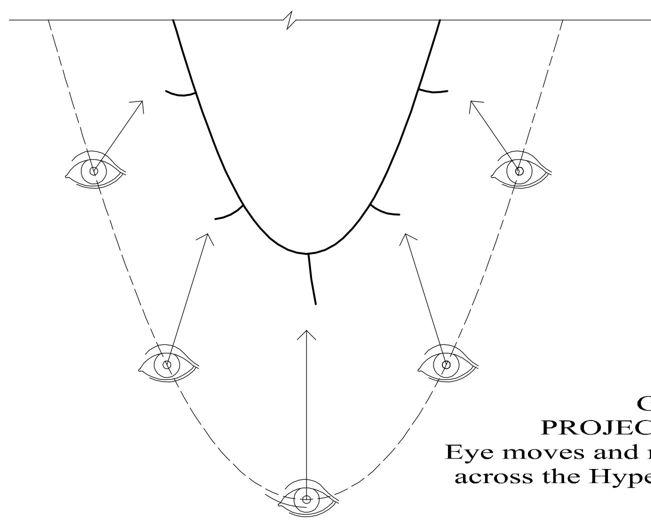
PERSPECTIVE uses a fixed Eye

GLIDE PROJECTION uses a moving Eye.

It is difficult to illustrate good views of flat planes in Hyperbolic Geometries. Their images diverge away from the Eye. It occurs to me that some sort of GLIDE PROJECTION system might inspire the invention of future Hyperbolic views taken from a roving Eye, or some sort of "concave fly's-eye" multi-Eyed camera.



PERSPECTIVE of a Hyperbolic Plane



GLIDE PROJECTION Eye moves and rotates across the Hyperbolic Plane

This book contains no GLIDE PROJECTION illustrations.

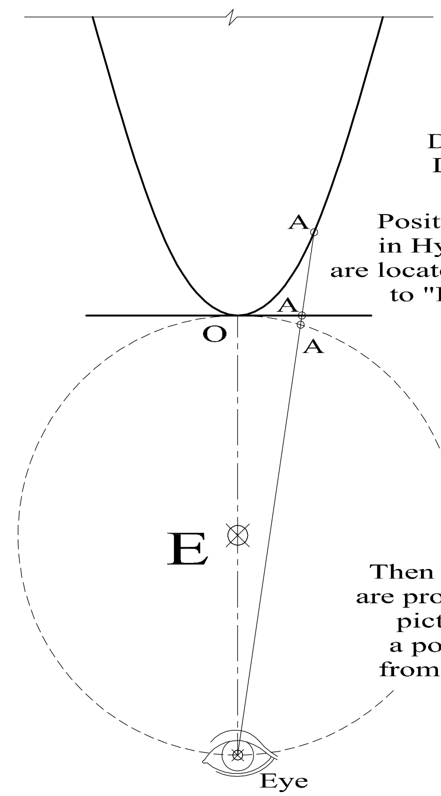
---

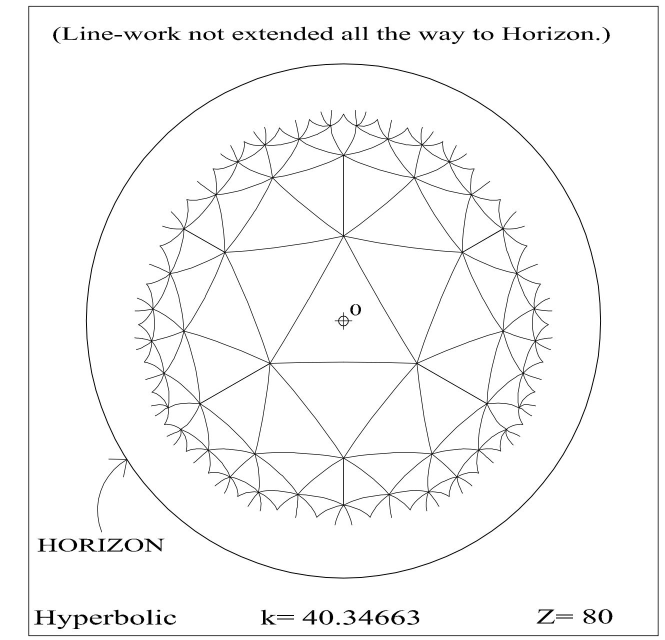
4. OTHER PROJECTIONS:

An endless variety of projection illustration methods might be imagined. Here, in *DRAWING 85*, is an unusual system where the positions of the Hyperbolic plane are established from point E, but the visual rays are projected onto the sphere from an Eye located at twice the perpendicular distance from the plane (on the sphere). This is not exactly a SPHERICAL PERSPECTIVE because the Eye is not at the center of the spherical picture plane; so, here I'm calling it a STEREOGRAPHIC PERSPECTIVE.

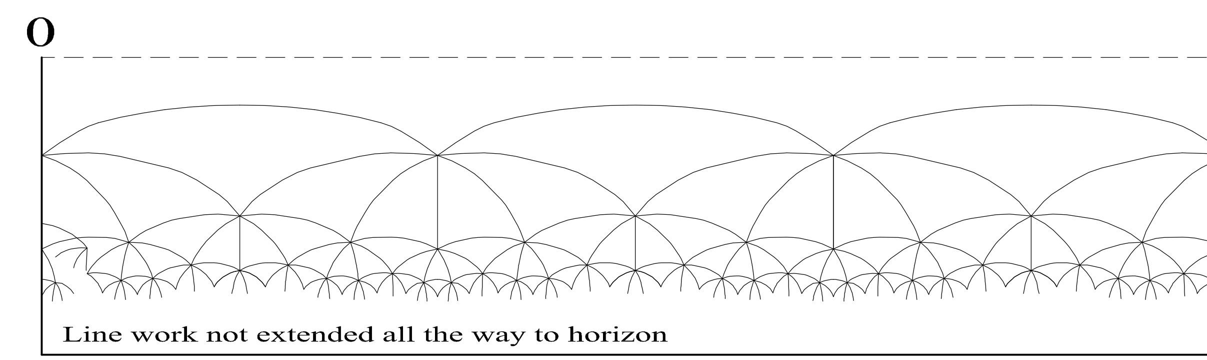
There are endless other possibilties for projection methods.

DRAWING 85: "STEREOGRPHAPHIC PERSPECTIVE" view of the tiling of equilateral triangles on a Hyperbolic plane.



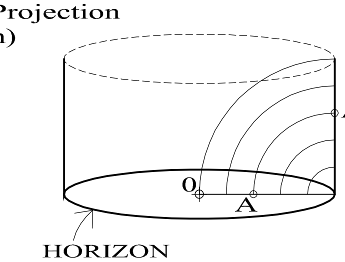


AZIMUTHAL STEREOGRAPHIC PROJECTION



CYLINDRICAL PROJECTION

Derived from the above Azimuthal Stereographic Projection by an Equidistant method (as shown in the diagram)

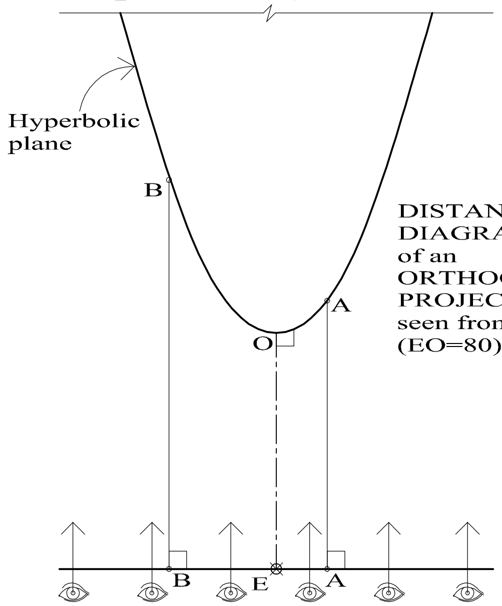


5. ORTHOGONAL PROJECTION:

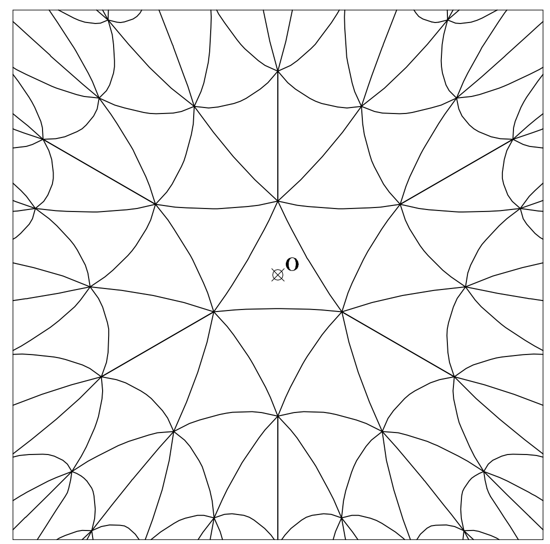
ORTHOGONAL PROJECTION, also called ORTHOGRAPHIC PROJECTION, is a method of projecting points from objects onto a flat picture plane along straight lines perpendicular to that picture plane.

As our examples we again consider the flat plane with tilings of equilateral triangles, set at a distance of 80.

In Non-Euclidean Geometries an ORTHOGONAL PROJECTION (taken from an assumed distance away from point "O") is slightly different from a Distance Diagram (taken directly from point "O").

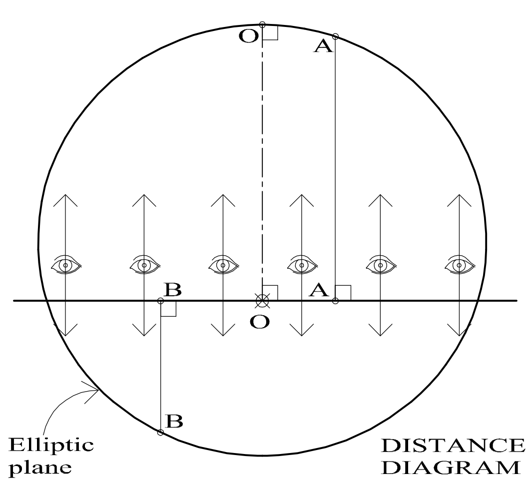


DISTANCE DIAGRAM of an ORTHOGONAL PROJECTION seen from point E (EO=80)

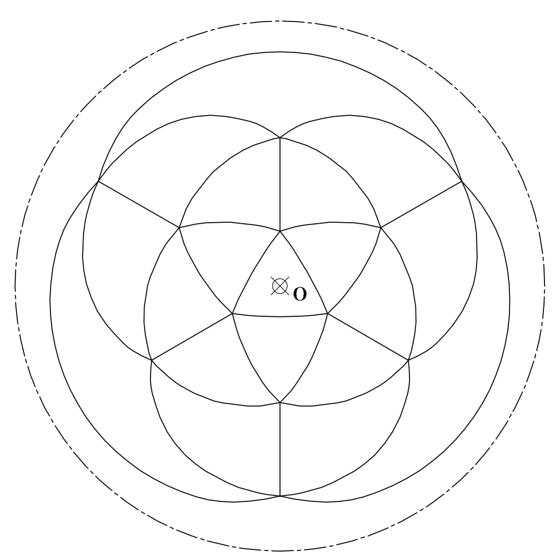


DISTANCE DIAGRAM seen from point "O"
Hyperbolic k= 40.3466

HYPERBOLIC



DISTANCE DIAGRAM of an ORTHOGONAL PROJECTION seen from point E (EO=80)

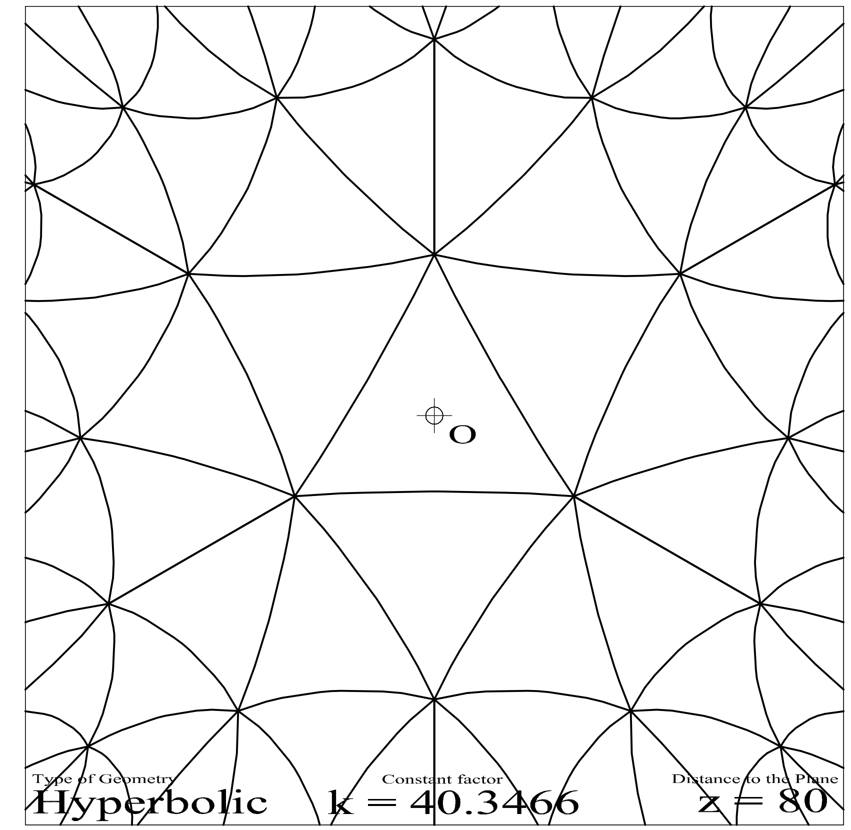


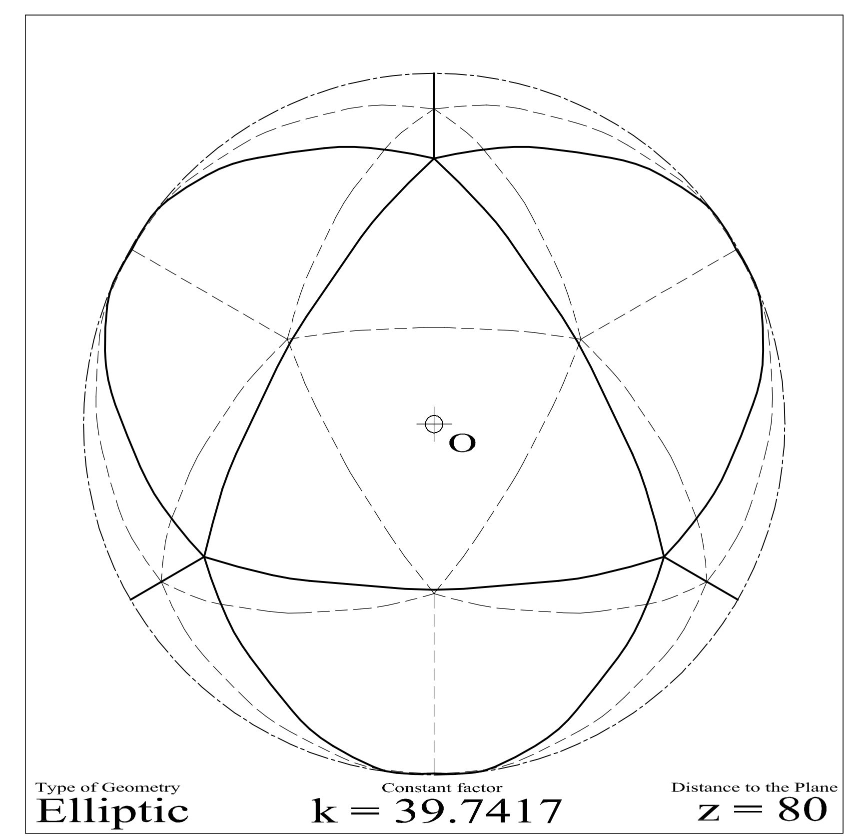
DISTANCE DIAGRAM seen from point "O"
Elliptic k= 39.7417

ELLIPTIC

DRAWING 86: **ORTHOGONAL PROJECTIONS**

Diagrams showing a tiling of equilateral triangles on a plane in Elliptic and Hyperbolic spaces.





In the business world of commercial drafting, ORTHOGONAL PROJECTION is the most commonly used method of visualization. "Plans, Sections, and Elevations" are Orthogonal Projections used for building construction.

I would like to show you a system of sketching with ORTHOGONAL PROJECTION that evolved as I drew little figures while trying to orgnize the calculations for the Perspective drawings in this book.

Not being agile in the use of advanced Non-Euclidean theorems, my calculation strategy simply triangluated the space into a network of right-angled-triangles, plus a very few non-right-angled triangles. Solving for points assigned to complex positions in 3-dimensional space quicky left with me an incomprehensible tangle of triangles -- it was hard to see what was, or was not, going to be a right-angle perpendicular in Non-Euclidean space.

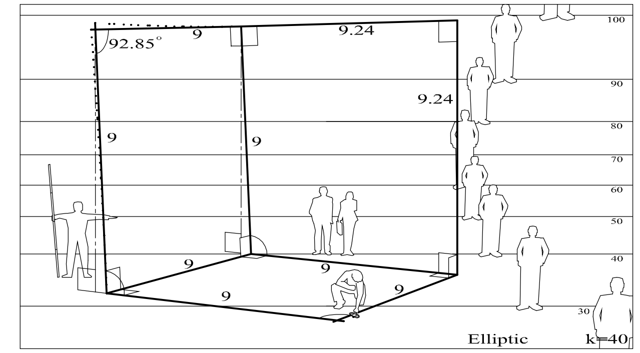
Sketching in an Orthagonal Projection format gave me a way to organize the right-angles and mentally hold onto them long enough to dispel utter confusion.

In its pure form an Orthogonal Projection drawing of Non-Euclidean Geometry would use a Distance Diagram as its base plane, but I preferred to keep straight lines straight (when mentally possible). So, my base plane figure is a sort of Perspective sketch of a plane slicing through the Non-Euclidean space. The alignments of points above and below the base plan are strictly orthogonal -- all the positions along a line perpendicular to the base plane will be drawn coinciding at the same point (on the base plane).

To draw on the picture plane the Perspective position of any point, the polar coordinates of its angles with respect to the Eye are needed. (I also calculate its distance, to sometimes use in further calculation steps or in "checking" computations). Though I initially used "X, Y, Z" (Cartesian) coordinates to compute Perspectives, I've discovered that polar coordinates are a bit faster.

For some people, solving Perspective geometry problems is a lot of fun. For learning a subject, the experience of solving problems for oneself is often unsurpassed. At this early stage of development, I find is easy to believe that such practices might find (or create) new and better methods to illustrate Non-Euclidean Geometry.

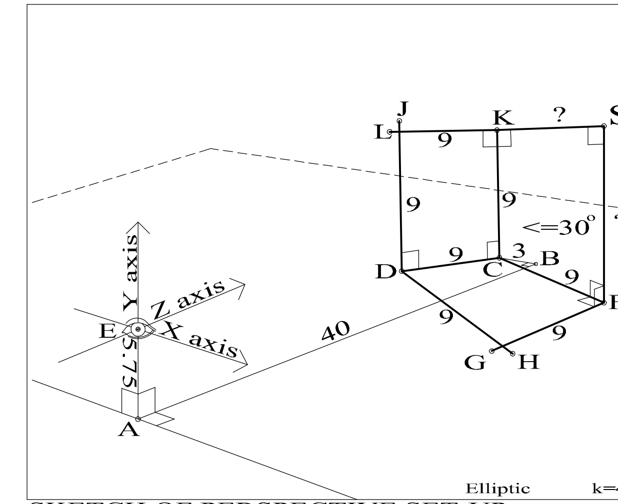
***DRAWING 87:* Quasi- ORTHOGONAL PROJECTION method used for sketching trianglulation of the location of one point ("S") of a complicated geometric figure in Elliptic space**



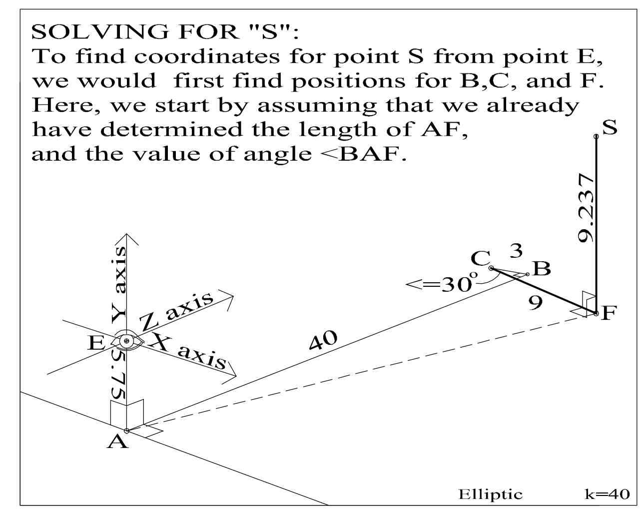
THREE "LAMBERT QUADRILATERALS"

Arrangement II -- 2 interior right angles meeting, and one exterior right angle

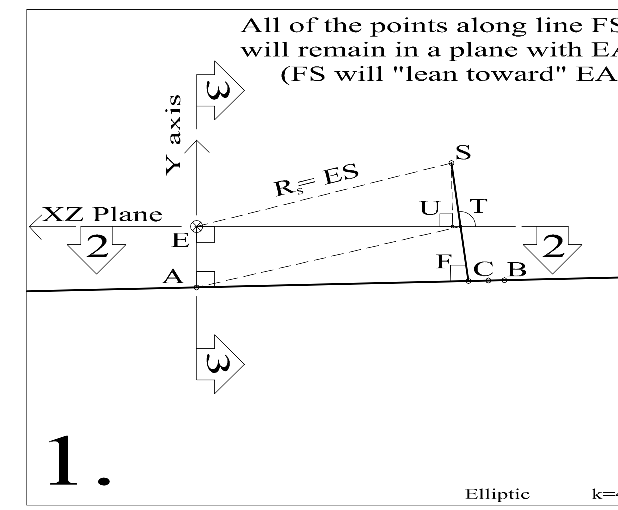
(Previously seen as ***Drawing 56* on** page 115)



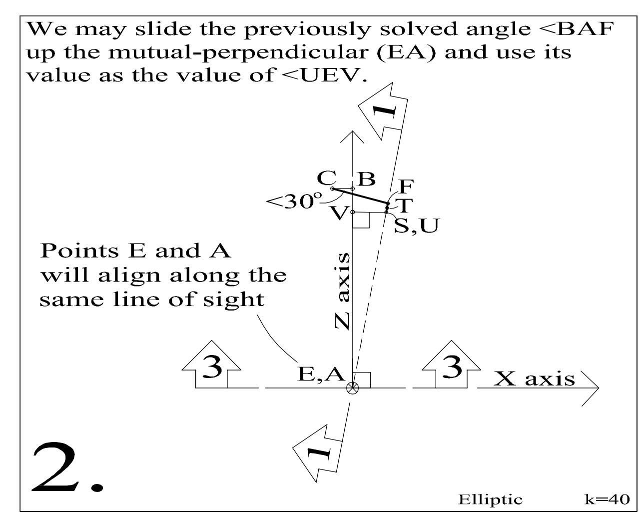
SKETCH OF PERSPECTIVE SET-UP



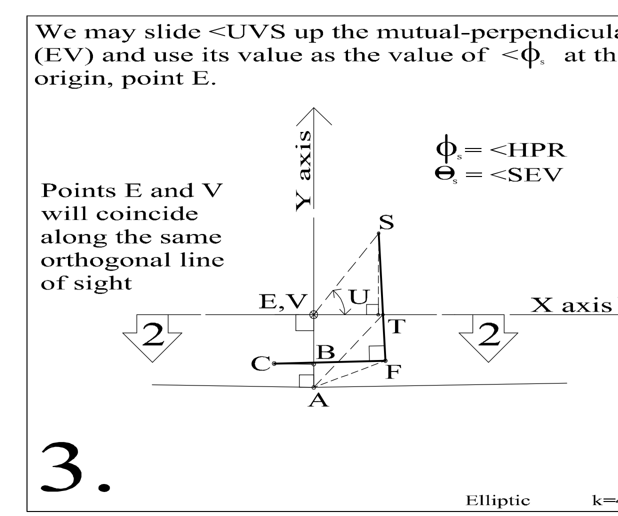
SKETCH SIMPLIFIED FOR POINT "S"



ORTHOGRAPHIC PROJECTION, SKETCH:
VIEW THROUGH THE PLANE OF "E,A,F,S"



ORTHOGRAPHIC PROJECTION, SKETCH:
VIEW THROUGH THE PLANE OF "E,T,U,V"--
A PLANE FORMED BY THE X AND Z AXES



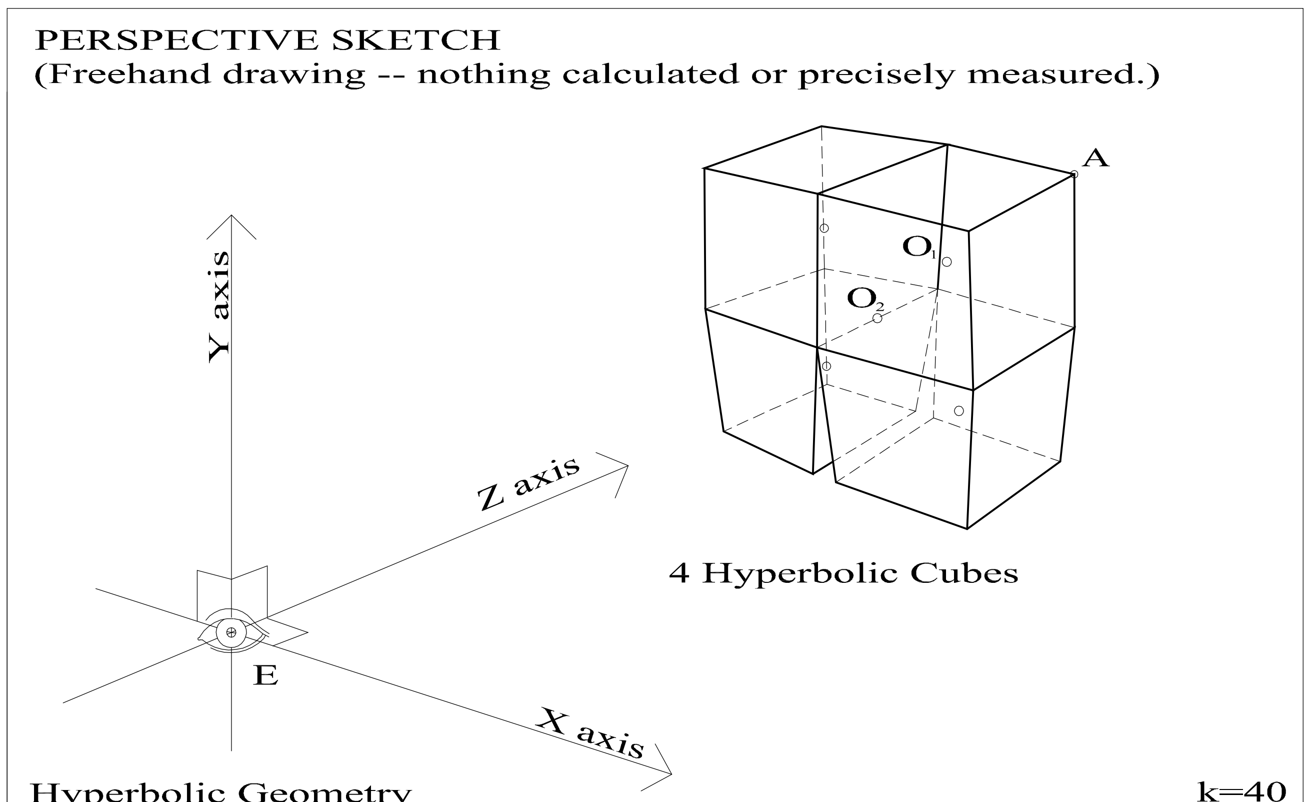
ORTHOGRAPHIC PROJECTION, SKETCH:
VIEW THROUGH THE PLANE OF "E,A" --
A PLANE FORMED BY THE X AND Y AXES

Slowly I tired of calculating each individual point all the way back to the position of the Eye; and started looking for faster procedures.

For quite some time I tried to formulate a way of transposing orthogonal coordinate axes from one point in Non-Euclidean space to another -- a generalized 'gyroscopic co-ordinate system' where positions of orthogonal axes could be reformulated from one place to another.

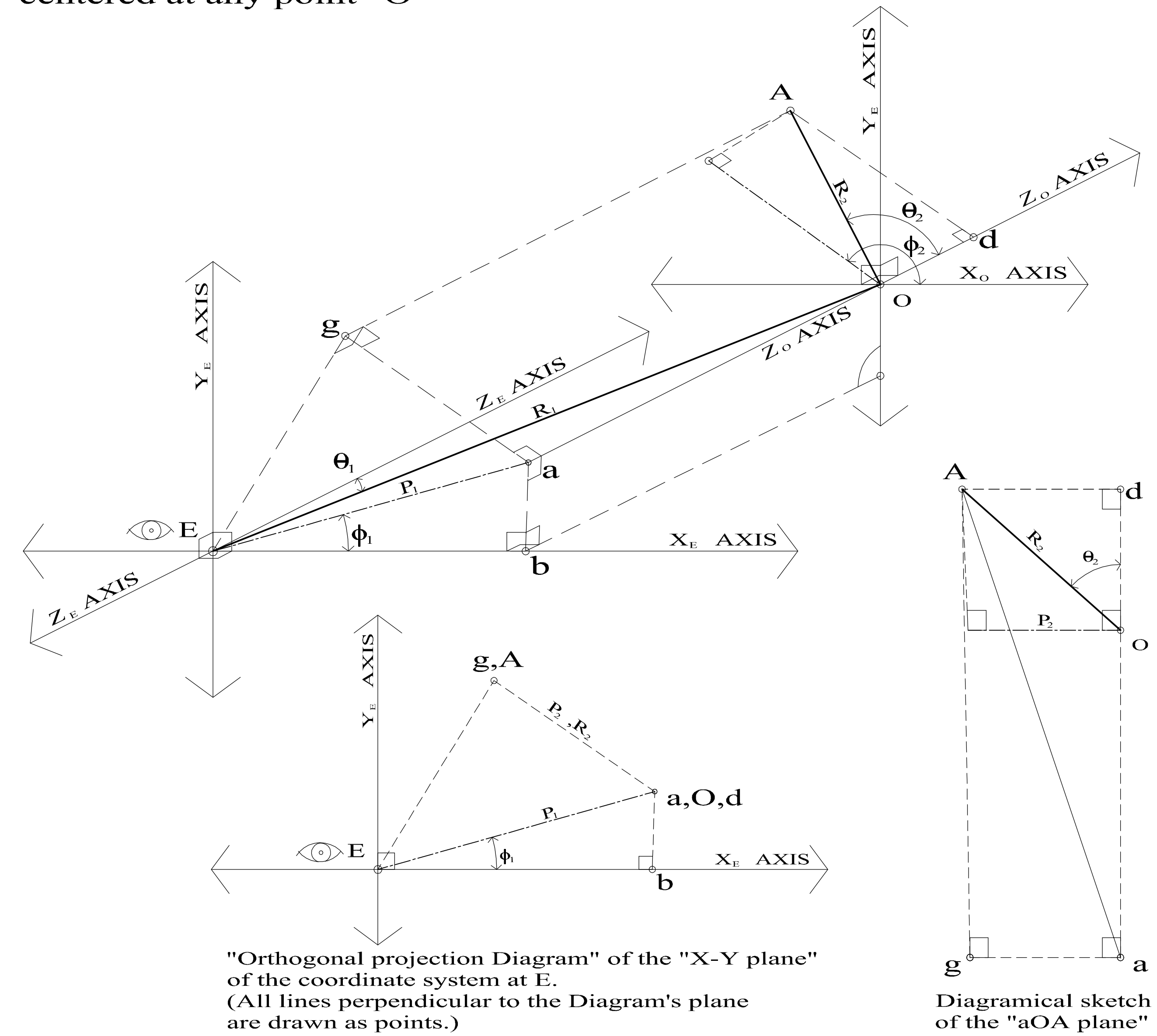
One of the persistent problems was the inability of my computer to differentiate which quadrant a trigonometric function was describing. Generalized procedures needed to crisscross between various 'quadrants' and across the poles of Elliptic space.

A calculation apparatus I developed depends heavily on ORTHOGONAL PROJECTIONS. I can (relatively) easily "look through the plane of the Eye" and quickly see into which quadrant a point should fall (then tinker with my computer-equations when necessary).



I can compute the postions of the corners of cubes with respect to point Bs, or as a chain of calculations from a point such as C. Rotations of a cluster in Non-Euclidean space can be at point "O" by Euclidean methods. This sped up my calculations, but I never got a fully mobile gyroscopic coordinate transformation system to work.

DRAWING 88: Solving for any point "A" in a coordinate system centered at any point "O"



Multiple orthogonal co-ordinate systems:
The Perspective is computed for a general point "A" seen from the Eye, at point "E".

Various points "A" of an object may be constructed in a co-ordinate system centered at point "O", then projected into Perspective from a second coordinate system centered at "E".

Resolving quadrant conditions for the various trigonomic functions is a headache, and I found that I could more easily envision this assembly using a planar Orthogonal Projection diagram, where the X-Y plane of the co-ordinate system centered at E gives views somewhat similar to the final image on the Perspective picture plane.

In the literature of mathematics there is already much about the following five models for visualizing Hyperbolic Geometry:

1. The ***HYPERBOLOID MODEL*** (also known as the *Weierstrass Model*, the *Minkowski Model*, or the *Minkowski-Lorentz Model*).

Images derived from it, by various projection methods, are:

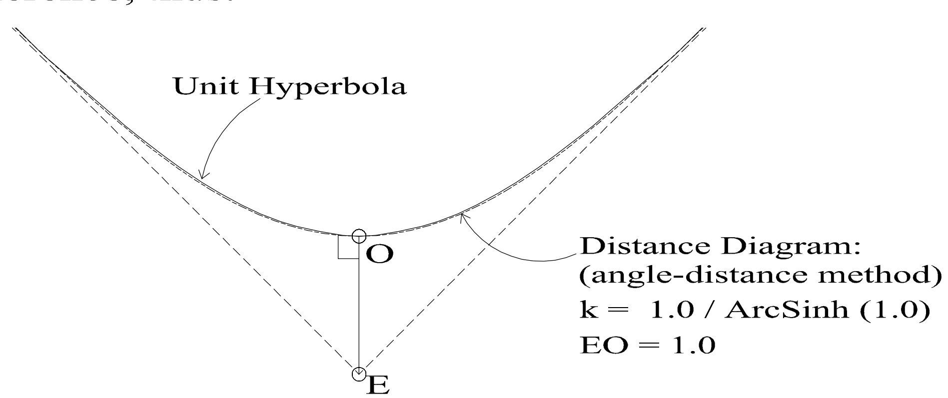
2. The ***KLEIN DISK MODEL*** (also known as the *Beltrami Model,* the *Beltrami-Klein Model,* the *Projective Model*, the *Cayley-Klein Model*, or *Central Projection);*
3. The ***POINCARE DISK MODEL*** (the *Conformal Disk Model);*
4. The ***POINCARE HALF-PLANE MODEL***; and
5. The ***GANS MODEL*** (Orthogonal, or Orthographic, Projection)

While constructing the Non-Euclidean Perspective drawings (and other visualizations) of this book, I was aware of these five models, but did not derive my methods directly from them. Because they looked similar, I imagined that the Hyperboloid Model was simply one of my Distance Diagrams (pages 154-155).

But now, at the end of my study, when I look closely at these models, I am surprised to see that I was wrong; the Hyperboloid Model is NOT a Distance Diagram, and therefore the Klein Model is (*apparently*) not exactly a Perspective method.

One obvious difference is that one of our Perspective assumptions, that the camera be relatively infinitismal with respect to the object being viewed, is violated by the set-up of the Poincare Models, whose relationship to the Klein Model depends on the sphere having a radius equal to the full distance to the object.

The more basic problem is that I have not been able to fit the Hyperboloid Model to a Distance Diagram. They look awfully similar, but I conclude that the Distance Diagram of a Hyperbolic plane is *apparently* NOT a hyperbola. The closest match I could find left a slight difference, thus:

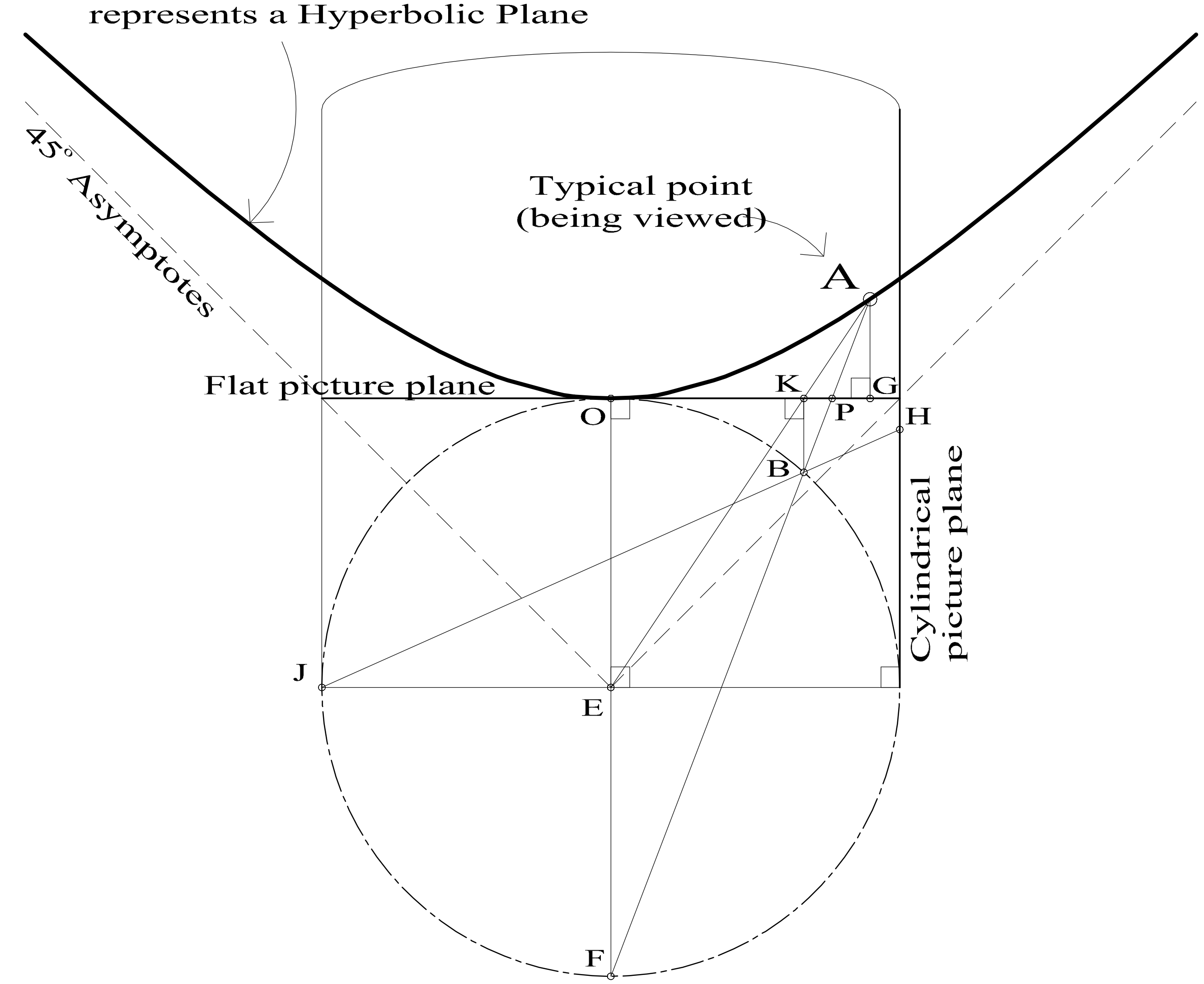


So, while the Unit Hyperbola of the Hyperboloid Model is related to the formulation of the hyperbolic trigonometric functions used in Hyperbolic Geometry, how that model represents a plane in a Hyperbolic space (and how the four subsequent projections relate to the standard methods of Descriptive Geometry) remains a problem.

13th January 2017

DRAWING 89: FIVE MODELS OF HYERBOLIC GEOMETRY
Euclidean "Section view", showing construction of all five models.

1. HYPERBOLOID MODEL:
a "Unit Hyperbola" (shown here) rotated about its cental axis, represents a Hyperbolic Plane



2. KLEIN MODEL: point "K" on the flat picture plane, projected from "E".

3. POINCARE MODEL: point "P" on the flat picture plane, projected from "F".

4. POINCARE HALF-PLANE MODEL: point "H" on a cylindrical picture plane projected from "J", through "B"

5. GANS MODEL: point "G" on the flat picture plane, projected orthogonally

WHY STUDY NON-EUCLIDEAN PERSPECTIVE ?

My friends ask, "What are you doing today?" and I sometimes describe this book. Then I'm asked, "Why are you doing that? What's the practical value?"

Two reasons NOT to study Non-Euclidean Perspective:

1. ... to replace Euclidean geometry.
Don't worry, your trusty old Euclidean Geometry isn't on its way out. To learn Non-Euclidean Geometries is to see why Euclidean Geometry was selected in the first place -- it's better balanced, more flexible, most elegant. Non-Euclidean Geometry is "expansion", not "replacement".

2. ... to learn Non-Euclidean Geometry.
Not exactly. If looking at Perspective pictures taught Euclidean Geometry then wasting time trying to teach Synthetic or Analytical Geometry by means of axiomatic logic would have ended long ago. A novice can learn quite a bit about an airplane by simply looking at pictures or movies; but building an airplane, or flying an airplane, requires deeper understanding.

Two reasons FOR studying Non-Euclidean Perspective:

1. Since the late 19th Century, there have been questions about how natural forces are transmitted invisibly through empty space. There has long been speculation that Non-Euclidean Geometries might possibly offer a flexible new model of Space, an aether which could deform (strain) in order to transmit force (stress) imposed upon it. Surprisingly similar to the mathematical descriptions of the elastic behavior of architectural structures, *The General Theory of Relativity* employs Non-Euclidean Geometry to describe the deformations of Space and Time caused by Gravity. Tying our visual brain faculties of sight into this study of the spatial deformations of Non-Euclidan spaces might perhaps someday lead to a better picture of other Force-transmissions.

2. To see what we can see.

PROBLEMS OF NON-EUCLIDEAN PERSPECTIVE

Once we unlock the Fifth Postulate of Euclidean Geometry, we get far too many new possibilities -- an infinity of Elliptic and Hyperbolic Geometries, Gausian Geometry, innumeral Topologies, Multi-Dimensional spaces, and so forth *ad nauseam*.

Likewise, once we start trying to visualize Non-Euclidean Geometry, we get not only traditional Perspective, but also Spherical (Curvilinear) Perspectives, Glide Projections, and the innumeral methods of Descriptive Geometry, (not to mention the personalized abstractions of Modern Art) -- far too many choices.

So, at the end of this book, may I please be permitted to suggest:

1. There is some sort of magic when straight lines appear on a picture plane as straight lines; and by that attribute alone we may single out Perspective (the Azimuthal Gnomonic Projection of Spherical Perspective) as our star. We should keep Perspective as our standard model of realism, even as we search for better methods to illustrate the peculiarities of Non-Euclidean spaces.
If discoveries in Neurology and Perceptual Psychology diminish our reliance on Perspective as the sole simultation of human eyesight, I expect Perspective nevertheless to advance as our "Ideal Vision" -- a mathematical generalization of the commonly seen comprehensive view of normal eyesight, and capable of having range and precision far beyond the limits of biological vision.

2. Computers change everything. There is a trend to stretch the Weistrass, Klein, Poincare, and Gans Models into more general forms of visualization. It is my guess that it would be better to preserve them in their original pre-computer states, and to expand into new computer-calculated visualizations of Non-Euclidean Geometry under the names already used in Descriptive Geometry.

3. In commercial drafting Orthgonal Projection methods are more commonly used than Perspective, and I see no reason why such shorter procedures (Distance Diagrams and Orthogonal Projections) should not serve similar utilitarian roles in Non-Euclidean work. But it is of paramount importance that viewers understand that it is Perspective alone that is our ultimate realistic view; and it seems safe to expect mental confusion if novices are introduced to diagrammatic illustrations without first having Perspective established as their standard of visual realism.

4. For any visualization, regardless of how standard or how unique, it is its ability to activate and sustain the imagination of its viewer (their inner-eye's grasp and understanding) which remains its highest purpose.

Appendix A:
Mathematical Formulae

To thoroughly understand Perspective visualizations of Non-Euclidean Geometry, one of the best learning methods is to construct your own Perspective images.

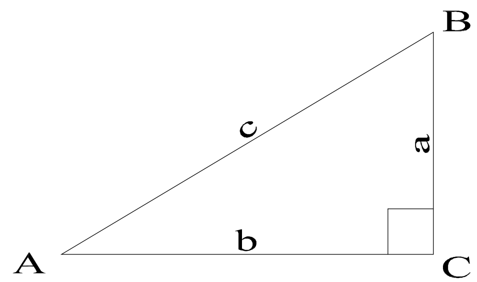
The images of this book used MicroSoft ***Excel*** computer bookkeeping software to compute points, which were then connected with lines and curves during drafting. The Excel 2011 version I used could figure values of inverse hyperbolic trigonometric functions (such as "ArcSinH"). All the equations I used are listed on the following two pages.

The Excel software was not primarily designed for Non-Euclidean calculations and I had some problems with its proper reading of the quadrant of the circle the angles were occurring in, and the rounding-off of some extreme numbers. But it was readily available software and worked reasonably well. Of course other computer methods will also work.

My images were drafted using ***AutoCad Lite 2000***.

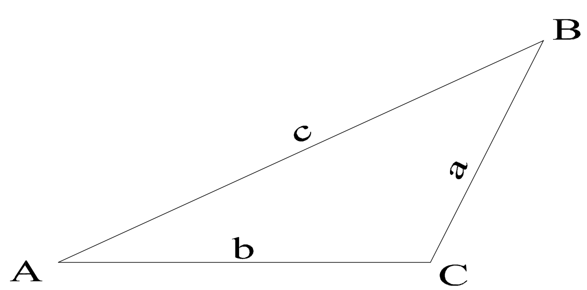
Trignometric Equations on a plane with invariant values of "k".

ELLIPTIC GEOMETRY



Relations of a RIGHT TRIANGLE

$\sin \mathbf{A} = \dfrac{\sin (\mathbf{a}/k)}{\sin (\mathbf{c}/k)}$	$\cos (\mathbf{c}/k) = \cos (\mathbf{a}/k) \cos (\mathbf{b}/k)$
$\sin \mathbf{B} = \dfrac{\sin (\mathbf{b}/k)}{\sin (\mathbf{c}/k)}$	$\cos (\mathbf{a}/k) = \dfrac{\cos \mathbf{A}}{\sin \mathbf{B}}$
$\cot \mathbf{A} \cot \mathbf{B} = \cos (\mathbf{c}/k)$	$\cos (\mathbf{b}/k) = \dfrac{\cos \mathbf{B}}{\sin \mathbf{A}}$
$\tan \mathbf{A} = \dfrac{\tan (\mathbf{a}/k)}{\sin (\mathbf{b}/k)}$	$\cos \mathbf{B} = \dfrac{\tan (\mathbf{a}/k)}{\tan (\mathbf{c}/k)}$
$\tan \mathbf{B} = \dfrac{\tan (\mathbf{b}/k)}{\sin (\mathbf{a}/k)}$	$\cos \mathbf{A} = \dfrac{\tan (\mathbf{b}/k)}{\tan (\mathbf{c}/k)}$



Relations of an OBLIQUE TRIANGLE

$$\cos (\mathbf{a}/k) = \cos (\mathbf{b}/k) \cos (\mathbf{c}/k) + \sin (\mathbf{b}/k) \sin (\mathbf{c}/k) \cos \mathbf{A}$$

(Beware the change of + to - signs between Hyperbolic and Elliptic Geometries)

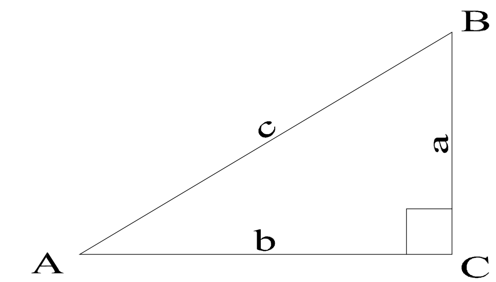
Euclidean: $a^2 = b^2 + c^2 - 2bc(\cos A)$

$$\sin (\mathbf{a}/k) : \sin (\mathbf{b}/k) : \sin (\mathbf{c}/k) = \sin \mathbf{A} : \sin \mathbf{B} : \sin \mathbf{C}$$

Euclidean: sin B: sin B: sin C = a: b: c

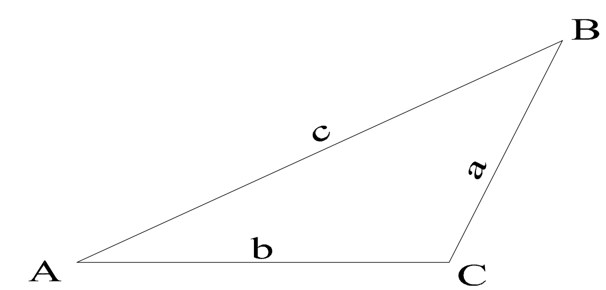
Trignometric Equations on a plane with invariant values of "k".

HYPERBOLIC GEOMETRY



Relations of a RIGHT TRIANGLE

$\sin \mathbf{A} = \dfrac{\sinh (\mathbf{a}/k)}{\sinh (\mathbf{c}/k)}$	$\cosh (\mathbf{c}/k) = \cosh (\mathbf{a}/k) \cosh (\mathbf{b}/k)$
$\sin \mathbf{B} = \dfrac{\sinh (\mathbf{b}/k)}{\sinh (\mathbf{c}/k)}$	$\cosh (\mathbf{a}/k) = \dfrac{\cos \mathbf{A}}{\sin \mathbf{B}}$
$\cot \mathbf{A} \cot \mathbf{B} = \cosh (\mathbf{c}/k)$	$\cosh (\mathbf{b}/k) = \dfrac{\cos \mathbf{B}}{\sin \mathbf{A}}$
$\tan \mathbf{A} = \dfrac{\tanh (\mathbf{a}/k)}{\sinh (\mathbf{b}/k)}$	$\cos \mathbf{B} = \dfrac{\tanh (\mathbf{a}/k)}{\tanh (\mathbf{c}/k)}$
$\tan \mathbf{B} = \dfrac{\tanh (\mathbf{b}/k)}{\sinh (\mathbf{a}/k)}$	$\cos \mathbf{A} = \dfrac{\tanh (\mathbf{b}/k)}{\tanh (\mathbf{c}/k)}$



Relations of an OBLIQUE TRIANGLE

$$\cosh (\mathbf{a}/k) = \cosh (\mathbf{b}/k) \cosh (\mathbf{c}/k) - \sinh (\mathbf{b}/k) \sinh (\mathbf{c}/k) \cos \mathbf{A}$$

(Beware the change of + to - signs between Hyperbolic and Elliptic Geometries)

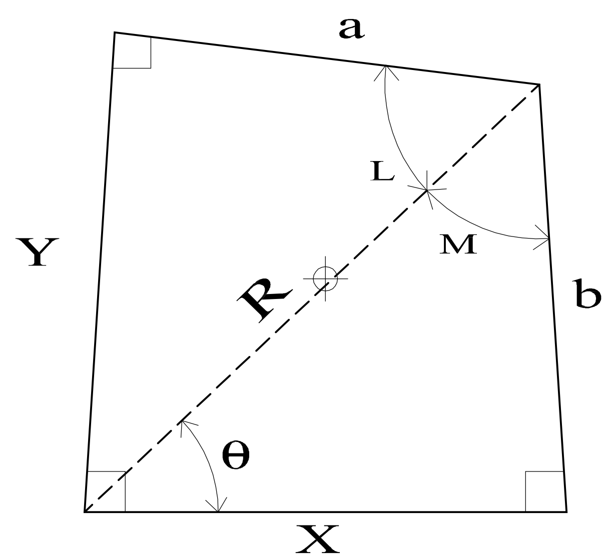
Euclidean: $a^2 = b^2 + c^2 - 2bc(\cos A)$

$$\sinh (\mathbf{a}/k) : \sinh (\mathbf{b}/k) : \sinh (\mathbf{c}/k) = \sin \mathbf{A} : \sin \mathbf{B} : \sin \mathbf{C}$$

Euclidean: sin B: sin B: sin C = a: b: c

Trignometric Equations on a plane with invariant values of "k".

ELLIPTIC GEOMETRY

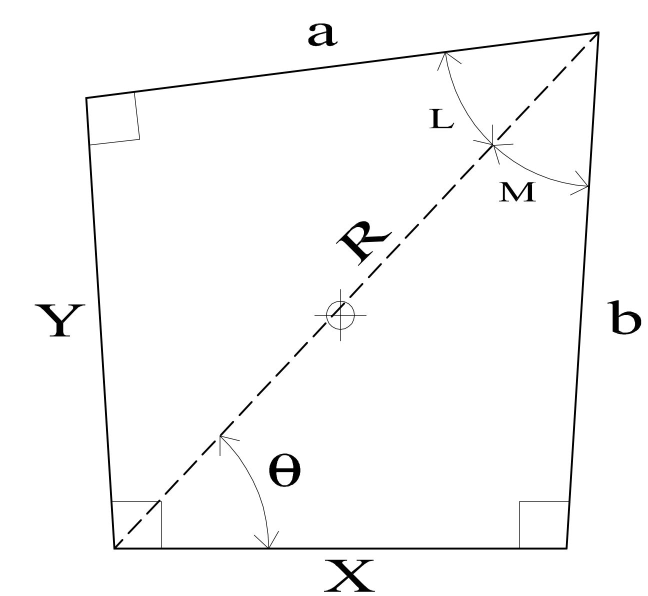


Relations of the "LAMBERT QUADRILATERAL"

$\tan \mathbf{X}/k = \cos \boldsymbol{\theta} (\tan \mathbf{R}/k)$	$\tan \mathbf{Y}/k = \sin \boldsymbol{\theta} (\tan \mathbf{R}/k)$
$\tan \boldsymbol{\theta} = (\tan \mathbf{Y}/k) / \tan \mathbf{X}/k$ $\tan \boldsymbol{\theta} = (\sin \mathbf{b}/k) / \sin \mathbf{a}/k$	$\sin L = (\sin \mathbf{Y}/k) / \sin \mathbf{R}/k$ $\sin M = (\sin \mathbf{X}/k) / \sin \mathbf{R}/k$
$(\tan \mathbf{R}/k)^2 = (\tan \mathbf{X}/k)^2 + (\tan \mathbf{Y}/k)^2$	
$\tan \mathbf{b}/\mathrm{k} = \sin \mathbf{X}/\mathrm{k} (\tan \boldsymbol{\theta})$	$\tan \mathbf{a}/k = \sin \mathbf{Y}/k (\tan \boldsymbol{\theta})$
$\sin \mathbf{b}/k = \sin \mathbf{R}/k (\sin \boldsymbol{\theta})$	$\sin \mathbf{a}/k = \sin \mathbf{R}/k (\cos \boldsymbol{\theta})$
$\cos \mathbf{b}/k = \cos \mathbf{R}/k / \cos \mathbf{X}/\mathrm{k}$	$\cos \mathbf{a}/k = \cos \mathbf{R}/k / \cos \mathbf{Y}/k$

Trignometric Equations on a plane with invariant values of "k".

HYPERBOLIC GEOMETRY

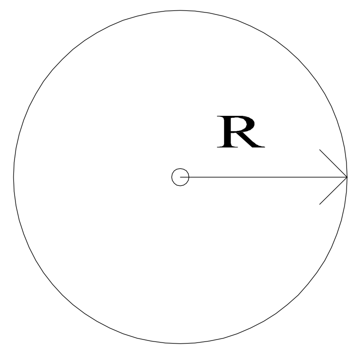


Relations of the "LAMBERT QUADRILATERAL"

$\tanh \mathbf{X}/k = \cos \boldsymbol{\theta} (\tanh \mathbf{R}/k)$	$\tanh \mathbf{Y}/k = \sin \boldsymbol{\theta} (\tanh \mathbf{R}/k)$
$\tan \boldsymbol{\theta} = (\tanh \mathbf{Y}/k) / \tanh \mathbf{X}/k$ $\tan \boldsymbol{\theta} = (\sinh \mathbf{b}/k) / \sinh \mathbf{a}/k$	$\sin L = (\sinh \mathbf{Y}/k) / \sinh \mathbf{R}/k$ $\sin M = (\sinh \mathbf{X}/k) / \sinh \mathbf{R}/k$
$(\tanh \mathbf{R}/k)^2 = (\tanh \mathbf{X}/k)^2 + (\tanh \mathbf{Y}/k)^2$	
$\tanh \mathbf{b}/k = \sinh \mathbf{X}/k (\tan \boldsymbol{\theta})$	$\tanh \mathbf{a}/k = \sinh \mathbf{Y}/k (\tan \boldsymbol{\theta})$
$\sinh \mathbf{b}/k = \sinh \mathbf{R}/k (\sin \boldsymbol{\theta})$	$\sinh \mathbf{a}/k = \sinh \mathbf{R}/k (\cos \boldsymbol{\theta})$
$\cosh \mathbf{b}/k = \cosh \mathbf{R}/k / \cosh \mathbf{X}/k$	$\cosh \mathbf{a}/k = \cosh \mathbf{R}/k / \cosh \mathbf{Y}/k$

ELLIPTIC Geometry	EUCLIDEAN Geometry	HYPERBOLIC Geometry

CIRCLE



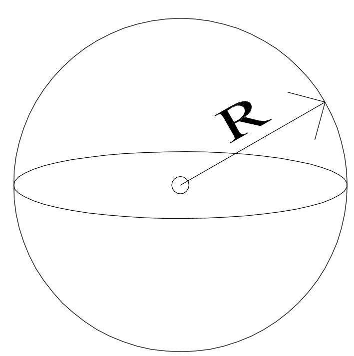
CIRCUMFERENCE of a planar Circle:

$= 2\,(\text{Pi})\,k\,(\sin R/k)$	$= 2\,(\text{Pi})\,R$	$= 2\,(\text{Pi})\,k\,(\sinh R/k)$

AREA of a planar Circle:

$= 2\,(\text{Pi})\,k^2(\sin R/k)^2$	$= 2\,(\text{Pi})\,R^2$	$= 2\,(\text{Pi})\,k^2(\sinh R/k)^2$

SPHERE



SURFACE AREA of a Sphere:

$= 2\,(\text{Pi})\,k^2(\sin R/k)^2$?	$= 4\,(\text{Pi})\,R^2$	$= 2\,(\text{Pi})\,k^2(\sinh R/k)^2$

VOLUME of a Sphere:

$=(\text{Pi})k^3(2R/k-\sin 2R/k)$?	$= 4/3\,(\text{Pi})\,R^3$	$=(\text{Pi})k^3(\sinh 2R/k-2R/k)$?

Bibliography:

Bhumkar, Kedar; ***Interactive visualization of Hyperbolic geometry using the Weierstrass model***; M.S. Thesis; University of Minnesota; July 2006; 69 pages.

Coxeter, H.S.M.; ***Non-Euclidean Geometry***; (first edition;1942); Mathematical Association of America; sixth edition, 1998; 336 pages.

Euclid, ***The Thirteen Books of THE ELEMENTS***; (c. 300 BC); translated with introduction and commentary by Sir Thomas L. Heath (Second Edition, 1925) ; Dover Publications, Inc.; New York (1956, etc.); 3 volumes.

Euclid, ***OPTICS of Euclid***; (c. 300 BC); translated by Harry Edwin Burton, *Journal of the Optical Society of America*, Vol. 35, No. 5 (May 1945), pages 357-372.

Manning, Henry Parker; ***Introductory Non-Euclidean Geometry;*** (formerly titled ***Non-Euclidean Geometry***; Ginn & Company; 1901); unabridged and unaltered republication by Dover Publications, New York (1963); 95 pages.

Sommerville, Duncan M'Laren Young; ***The Elements of Non-Euclidean Geometry***; Open Court Publishing Co., Chicago; London: Bell & Sons Ltd. (1919); photo-reproduction edition published by Forgotten Books (2012); 274 pages.

Weeks, Jeffrey R.; ***The Shape of Space: How to Visualize Surfaces and Three-Dimensional Manifolds***; Marcel Dekker, Inc.; New York and Basel (1985); 324 pages.

Wolfe, Harold E.; ***Introduction to Non-Euclidean Geometry***; Holt, Rinehart, and Winston Inc., New York (1945); unabridged replication by Dover Publications, Mineola, New York (2012); 247 pages.

(End of Bibliography)

An Introduction to
the Perspective Illustration of Non-Euclidean Geometry

INDEX

INDEX

5th Postulate (The Fifth Postulate of Euclid's ***Elements***) 3-4, 86-89

Absolute, The........ (see: Horizon, and Horocyle)
Apparatus setup for making Perspective images (and alternatives to the setup of this book)...... 7-8, 156
Apparent (Note explaining the special meaning of the term in this book) 50
Approximately Euclidean space, definition of 5, 6
Azimuthal Projections of Spherical Perspectives 156-158

Barnes, James David (Jim Barnes, architect) 1
Bibliography........ 178
Boundary Curve, the .... (see: Horizon, and Horocyle)
Bumkar, Kedar: *Interactive Visualization of Hyperbolic Geometry using the Weierstrass Model*; 2006 178

Camera (or camera obscura), dimensions of ....(see: Apparatus)
Cartesian Coordinate system impossible in Non-Euclidean Geometries 18
Central Line of Sight (Central Ray of Vision) 8
Circles (Proper Circles) 50-51, 67-68
Circumference of Circles (varying in Non-Euclidean Geometry)........ 102-107
Conformal Disk Model .... (see: Poincare Disk Model)
Copyright stipulations 1
Cube (Hyperbolic) 118-123
Curved, or warped, space (discussion of terminology) 46

Density of space (varying in Non-Euclidean Geometries) 10
Distance Diagrams (definition of, and 3 methods of drawing) 154-155
Distance Diagrams of Elliptic Plane 42, 44, 81-84, 145-148

Elliptic Geometry,definition of 3,4
Equidistant Azimuthal projection (Spherical Perspective) 157-158
Equidistant Cylindrical Projection (Spherical Perspective) 159
Equidistant Curves 37-38, 69-80
Equidistant Curves (graphic construction of) 75-76
Equidistant Surfaces 139-148
Equidistant Tubes .... (see Equidistant Surfaces)
Equilateral Triangles, Tilings of 60-61, 128-135
Euclid (author of ***The Elements***, and ***Optics***) 178
Eye (Point of view, aperture), definition of 7, 8

Fifth Postulate .... (see: 5th Postulate)
Formulae for triangles, quadrilaterals, Circles and Spheres .... (see: Mathemtical Formulae)
Forseth, Kevin, author of ***Glide Projection: Lateral Architectural Drawing***; 1984 160

Gans Model (Orthogonal Projection, or Orthographic Projection) 168
Glide Projections (based on the book: ***Glide Projection*** by Kevin Forseth, 1984) 160
Gnomonic projection (or gnomonic perspective, the Gnomonic Azimuthal projection) 7, 157
Gnomonic Cylindrical Projection (Spherical Perspective) 159
Grubbs, Christopher (Perspective Illustrator) 3, 35
Gyroscopic Non-Euclidean Coordinate transfers (discussion of idea of) 166

Horizon (Hyperbolic Plane) 33-34, 108-109
Horizon, discussion about the optical appearance of Horizons on Earth 112-113
Horocyle (definition of) and Perspective image of 77
Horosphere 151-152
Human vision (eyesight) 7
Hyperbolic Geometry, definition of 3, 4
Hyperboloid Model (Weierstrass, Minkowski, or Minkowski-Lorentz Model) 168-169

Ideal Point (definition of) 64
Intersection of Flat Planes........ 126-127
Italian Renaisance Perspective........ 7

K factor (explained) 16
"k" factor, varying 58-59, 90-91
Klein Disk Model (Beltrami Model, Betrami-Klein Model, Central Projection) 168

Lambert Quadrilateral 58-59, 114-117
Line (straight line) 7
Lines, names for types of Lines 64-65
Limiting Curve, the .... (see: Horizon, and Horocyle)

Manning, Henry Park: (author of ***Introductory Non-Euclidean Geometry***) 178
Mathematic Formula for Non-Euclidean Geometry 172-177
Minkowski Model (or Minkowski-Lorentz Model)........ (see: Hyperboloid Model)

Non-Euclidean, definition of 3
Non-Euclidean Perspective, why study 170
Non-Euclidean Perspective, problems briefly discussed 171

Oricyles .... (see: Horocyle)
Orthogonal Projections of Non-Euclidean Geometry (Orthographic Projection)........ 162-167
Orthographic Azimuthal Projection (Spherical Perspective) 157
Orthographic Cylindrical Projection (Spherical Perspective) 159

Parallel Lines in Hyperbolic Geometry, definition of 64-65
Pentagons, right-angled tiling of in Hyperbolic space 62-63
Perspective apparatus .... (see: Apparatus)
Picture Plane (view window, drawing surface, veil), definition of 8
Pinhole Photography 7
Perspective, definition of 7
Planes, Perspective appearance of flat surfaces........ Chapter 3 (starting on page 22), 124-135
Poincare Disk Model (Conformal Disk Model) 168-169
Poincare Half-Plane Model 168-169
Poles in Elliptic Geometry (one-pole and two-pole versions) 79
Polygons (appearance of) 54-55
Polyhedrons, (Regular Polyhedrons, also called the Platonic Solids) 118-123, 158-159
Precisely Euclidean spaces, definition of 5, 6
Proper Circles .... (see: Circles)
Proper Spheres .... (see: Spheres)

Representative Line (definition of) 69-74
Right Angle, definition of 3

Saccheru Quadrilateral (introduced)........ 10
Size of Images (Elliptic Perspective images are larger, Hyperbolic smaller) 12, 98-101, 149-150
Similarity doesn't work in Non-Euclidean Geometry 14
Sketching in Perspective and in Orthogonal Projection 163-167
Spheres (Proper Spheres) 137-138
Spherical Perspective methods (Curvilinear Perspective) 156-159
Spherical Surfaces, varying areas in Non-Euclidean Geometry 32
Stereographic Azimuthal Projection (Spherical Perspective) 157
Stereographic Cylindrical Projection (Spherical Perspective) 159
Stereographic Perspective (shown in Azimuthal and Cylindrical Stereographic projections) 161
Straight Lines always appear Straight in Non-Euclidean Perspective 7, 52-53

Tiling of Equilateral Triangles ....(see: Equilateral Triangles, Tilings)
Triangles (sum of interior angles greater than, or less than, 180 degrees) 12, 94-97
Triangles, equilateral 56-57
Trigonometry for Hyerperbolic and Elliptic Geometry .... (see: Mathetmatical Formulae)

Ultra-Ideal Points (definition of) 75

Weeks, Jeffrey R. (***The Shape of Space***) 178
Weierstrass Model .... (see: Hyperboloid Model)
Wolfe, Harold E. (***Introduction to Non-Euclidean Geometry***) 178
Wood, Grant ("*The Midnight Ride of Paul Revere*") 35

(End of Volume)