Non-Euclidean Perspective

Part 3

An Introduction
to the
Perspective Illustration
of
Non-Euclidean Geometry

by

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Chapter 6:

Regarding the Perspective appearance of straight-line figures and flat surfaces arranged in 3-dimensional Non-Euclidean spaces

This chapter will discuss the Perspective visual appearance of straight lines, figures drawn with straight lines, and flat planes.

Typically, geometry is taught first in a two-dimensional manner, then a third dimension is added; but because of the peculiar visual appearance of Non-Euclidean planes, I think it might be easier for the reader to see 3-dimensional visualizations first. So, this chapter is set up to be used as the introduction to the visualization of simple Non-Euclidean geometries. Future re-organization might put this chapter after Chapter 1.

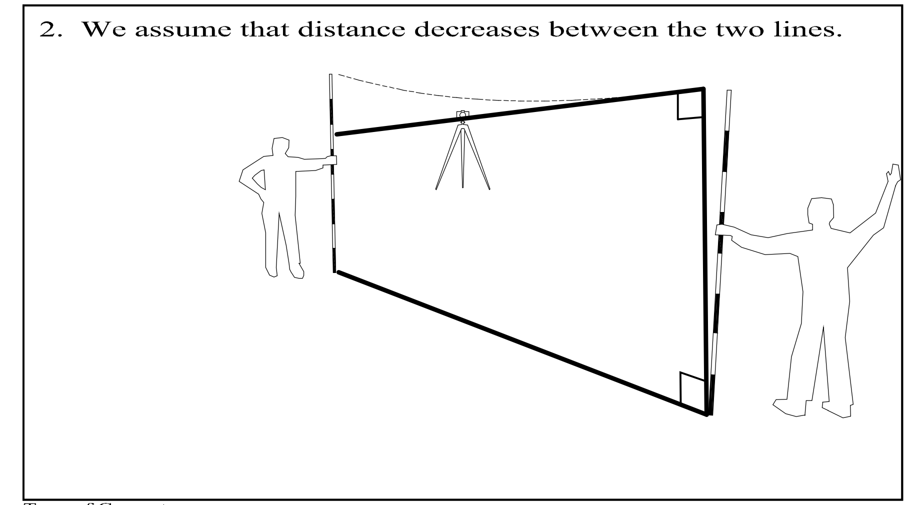
In this book, our Perspective picture-making set-up (our camera) will always be the same, except that sometimes the frame around the picture plane is enlarged into a square, (rather than using its normal rectangular shape). Otherwise, the distance and angle between the point we call the Eye (or pinhole aperture) and the flat Picture Plane onto which the image is projected, will never vary. The size of the picture image is thereafter sometimes enlarged or diminished to fit onto various page layouts. Other Perspective set-ups are certainly possible, but are not used in this book.

For *Drawing 41* we set up a base line with two longer co-planar perpendicular lines drawn from each end, projecting outward into the space. In all three geometries these figures are set up precisely the same -- distances and angles are all exactly the same. The only thing different in the three drawings is our variations to Euclid's famous 5th Postulate, our assumptions about the distance between those two co-planar perpendiculars.

In *Drawing 41* we may readily see that straight lines appear straight. (This turns out to be true always). The perpendiculars (also called "Right Angles", or "90° Angles"), appear pretty much the same in each different assumption. What appears to have changed is what we assumed would change: distance. The size of objects in the distance varies -- the space appears to be "compressing" or "expanding".

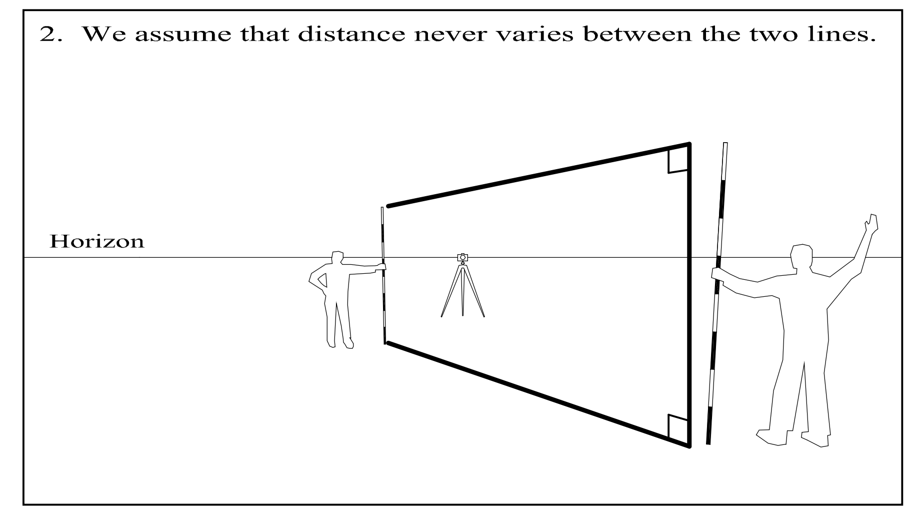
In the background I have set up a similar observer, a camera looking back from the opposite direction . . .

DRAWING 41: Two co-planar lines are set perpendicular to a vertical. A camera views them under 3 different assumptions.

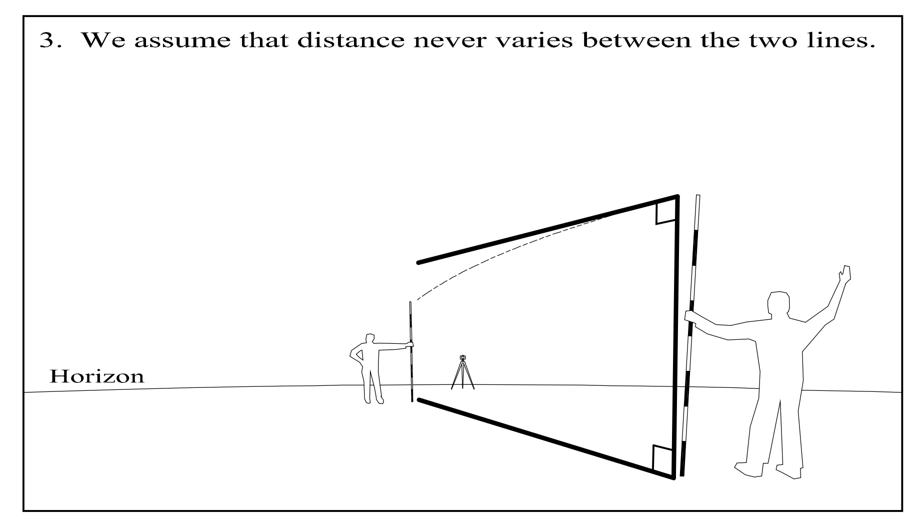


Type of Geometry
Elliptic

Constant factor
k = 50



Type of Geometry
Euclidean



Type of Geometry
Hyperbolic

Constant factor
k = 50

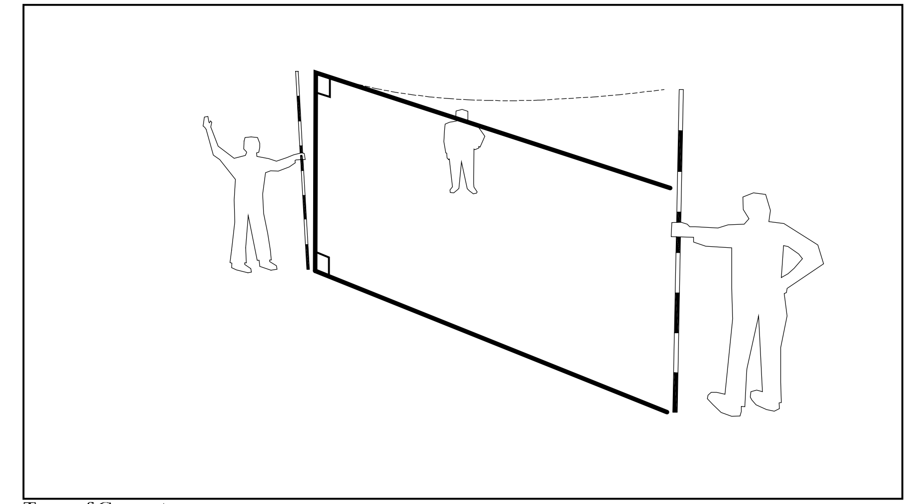
Non-Euclidean space can be described as a change in density. In Hyperbolic space, measured sizes seem to compress as we look outward. In an opposite manner, measured sizes (distances) seem to expand as we look outward into Elliptic space. Points, lines, angles, circles and planes do not change. It is the distance between the lines, within the fixed angles, that is changing. The density of "distance" (measured lengths) is the change we see.

These simple forms of Non-Euclidean geometry are completely homogenous, and symetrical. Every point in these space has exactly the same character as ever other point. More complicated forms of Non-Euclidean Geometry, where the "density of measured lengths" varies from place to place within the space, are possible but are not discussed in this book.

Explanations describing the Non-Euclidean space as "curving" or being "warped" do not seem to me to be appropriate. We will see shortly that flat planes will have the visual appearance of warping, but such curving is merely an optical effect (as we will see later in this chapter).

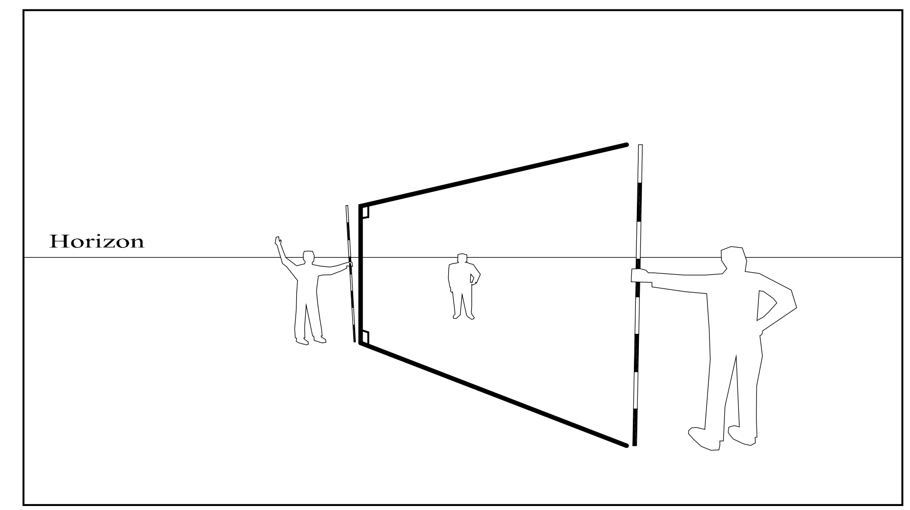
In Non-Euclidean Geometries, the rate at which "length" grows denser, or less dense, is regulated by the constant factor "k". The next drawing shows what happens at various different values for "k".

DRAWING 42: **Looking back in the opposite direction. In the distance we see the position of the observer for *Drawing 41*.**

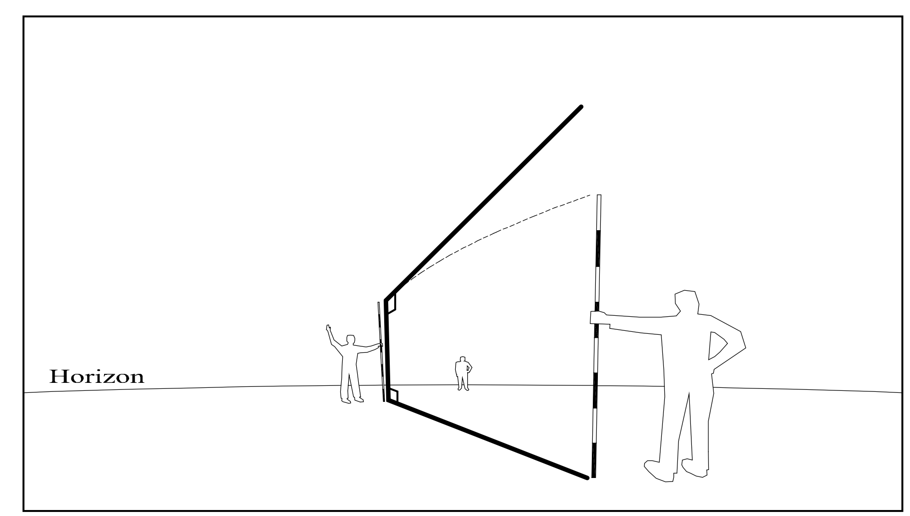


Type of Geometry
Elliptic

Constant factor
k = 50



Type of Geometry
Euclidean



Type of Geometry
Hyperbolic

Constant factor
k = 50

When values of "k" are very large, Hyperbolic and Elliptic spaces become approximately Euclidean. As "k" values become smaller, the Hyperbolic spaces become denser and the Elliptic spaces become less dense.

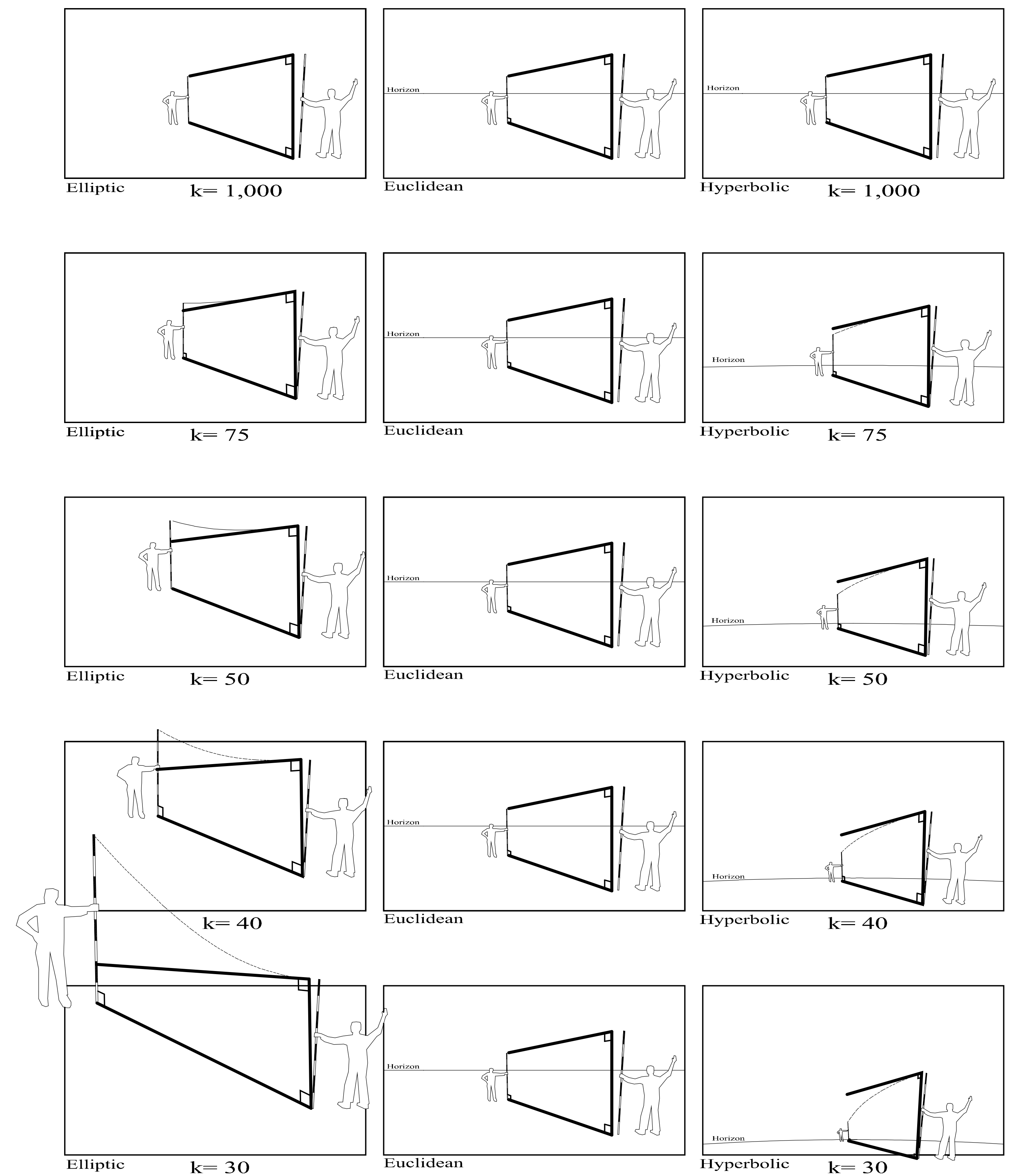
A couple of remarks:

When "k" becomes very big, a space becomes approximately Euclidean. In real life we can not ever be certain we are living in a precisely Euclidean world or whether the "k" value is simply a bit larger than our surveying tools are able to measure. To be totally precise, we can only say that we *assume* that we are in a Euclidean space.

Secondly, each "k" factor gives a Non-Euclidean Geometry a unique scale for measurements. Unlike Euclidean space, where a unit for measuring lengths must be arbitrarily invented, in Non-Euclidean spaces "k" can become a unversal measuring rod for that space.

The human figures are added to give a sense of size -- to graphically illustrate the manner in which 'distant' is getting denser, or less-dense. They are sketched approximately to scale, so that all the humans are the same size within each Drawing. If you wish to say that humans in these Non-Euclidean Perspectives are illogical, you may ignore them -- the line figures are logical without the decorative human figures.

DRAWING 43: **Same as *Drawing 41*, but varying "k".**



In *Drawing 44* we add two more lines to the three lines of *Drawing 42*. We add a third perpendicular at the mid-point of our upright base. And we draw a fourth line at a right angle to it, connecting the ends of the two outer perpendiculars. This closes the three lines we first saw in *Drawing 41* as two four-sided figures. It turns out that these two four-sided polygons are congruent mirrors of each other.

"Lambert Quadrilateral" is a name given to these four-sided planar figures. A Lambert Quadrilateral has four straight sides and right angles at three of its corners.

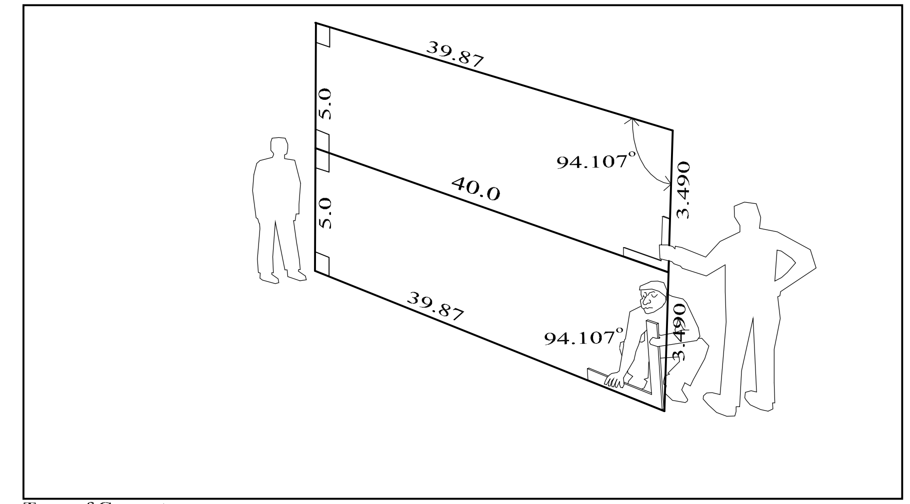
In all three geometries, the two Lambert Quadrilaterals on each side of the newly added middle line will always be mirrors of each other -- all the sides and angles will match to the corresponding elements on the opposite side.

It is a general rule that in Non-Euclidean Geometries, the fourth corner of a Lambert Quadrilateral will never be a right angle (though as 'k' gets very large, or the size of the figure gets very small, that fourth corner will approach a limiting value of 90°.)

In Elliptic spaces, the fourth corner will always be GREATER than a right angle; and the sides adjacent to that corner will alway have lengths SHORTER than their opposite sides.

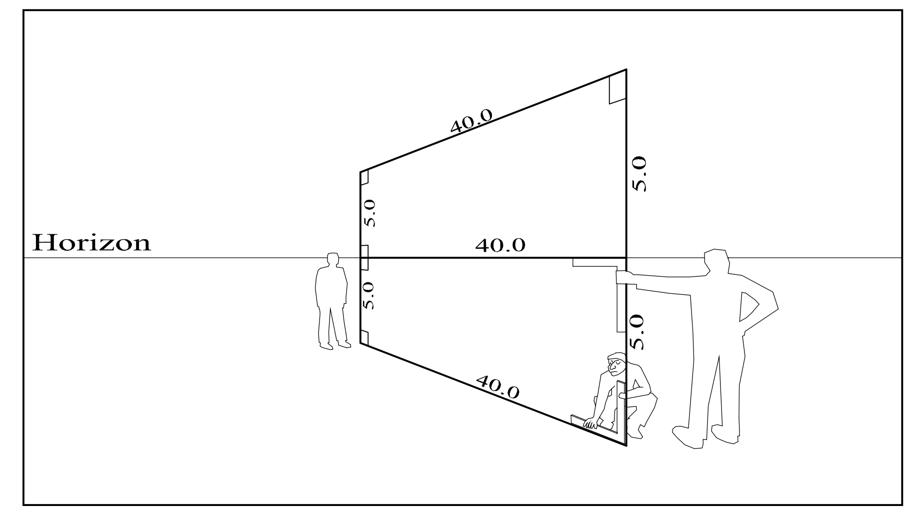
In Hyperbolic Geometries, the fourth corner will always be LESS than a right angle; and the sides adjacent to that corner will alway be LONGER than their opposites.

DRAWING 44: **Add a third co-planar perpendicular at the midpoint of the baseline, and fourth line at its end, set at 90°.**

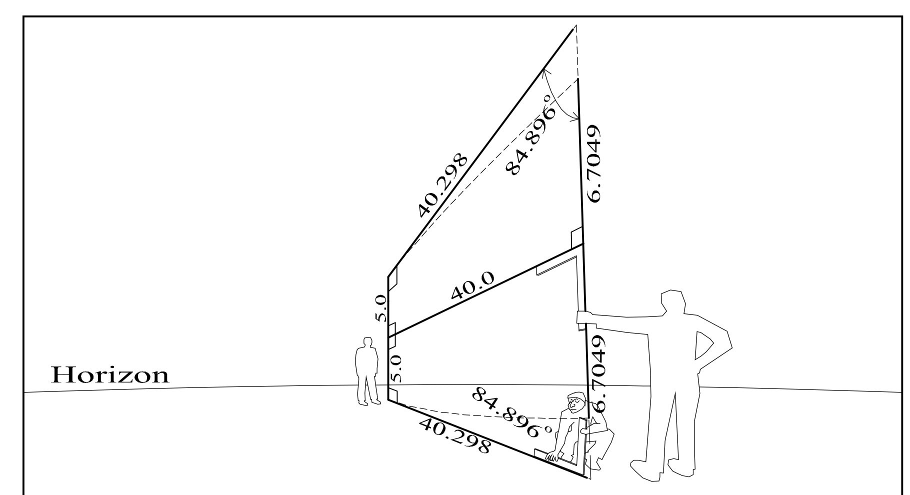


Type of Geometry
Elliptic

Constant factor
k = 50



Type of Geometry
Euclidean



Type of Geometry
Hyperbolic

Constant factor
k = 50

If we start dividing our 3-right-angled Lambert Quadrilaterals into triangles, and we measure the resulting elements, we find the following:

In Elliptic Geometry, the sum of the angles of a triangle wil always be GREATER than 180° (the sum of two right angles).

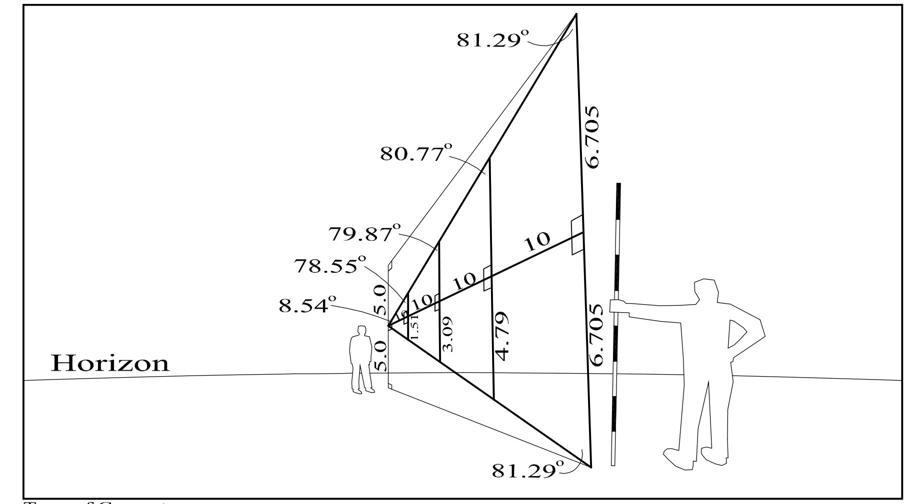
In Hyperbolic Geometry, the sum of the angles of a triangle wilL always be LESS than 180° (the sum of two right angles).

But as the value of "k" becomes infinitely large, or the size of the triangle becomes infinitely small, then the sum of its angles will approach 180°.

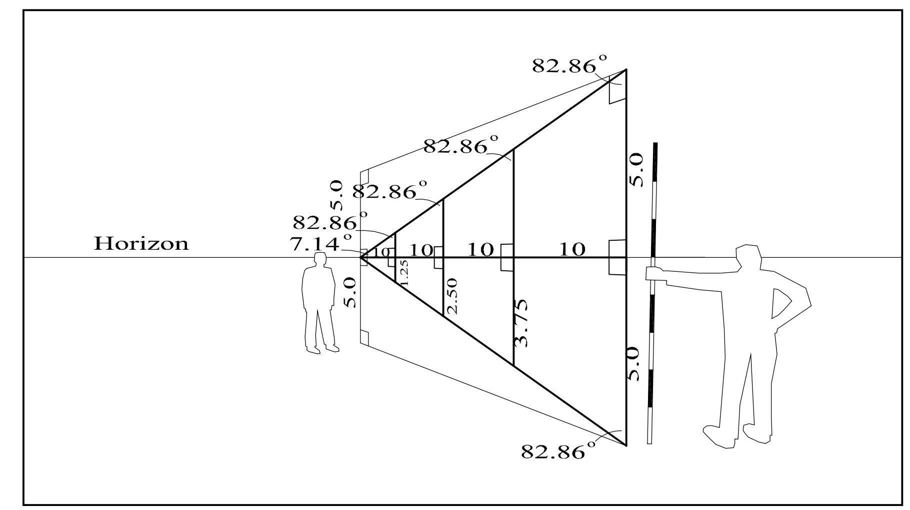
And as the size of the triangles get larger and larger, the sum of their three angles departs farther and farther from 180°.

The principle of SIMILARITY does not work in Non-Euclidean Geometry. Triangles can not be scaled up-or-down to different sizes without changing the measured values of their angles and the proportional lengths of their sides.

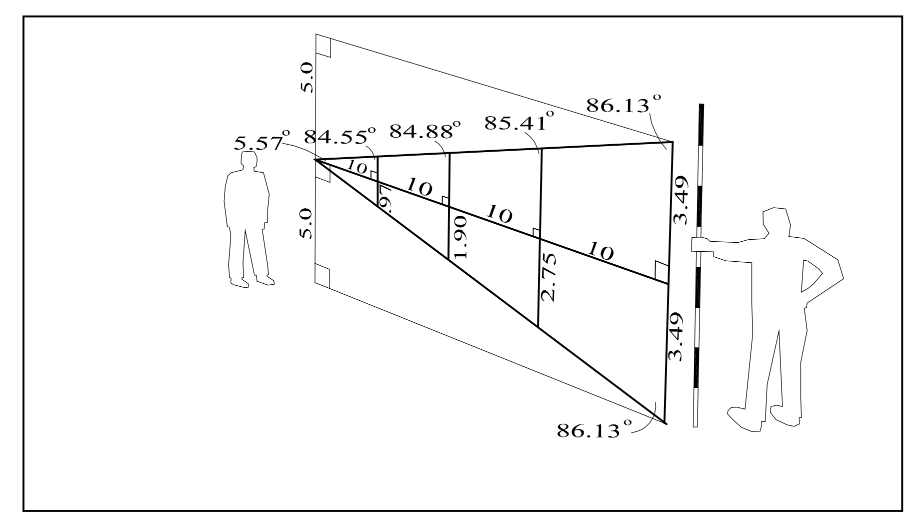
DRAWING 45: **Diagonals are drawn across both quadrilaterals. The mid-line is divided into 4 parts, with perpendicular legs.**



Type of Geometry
Elliptic
Constant factor
k = 50



Type of Geometry
Euclidean



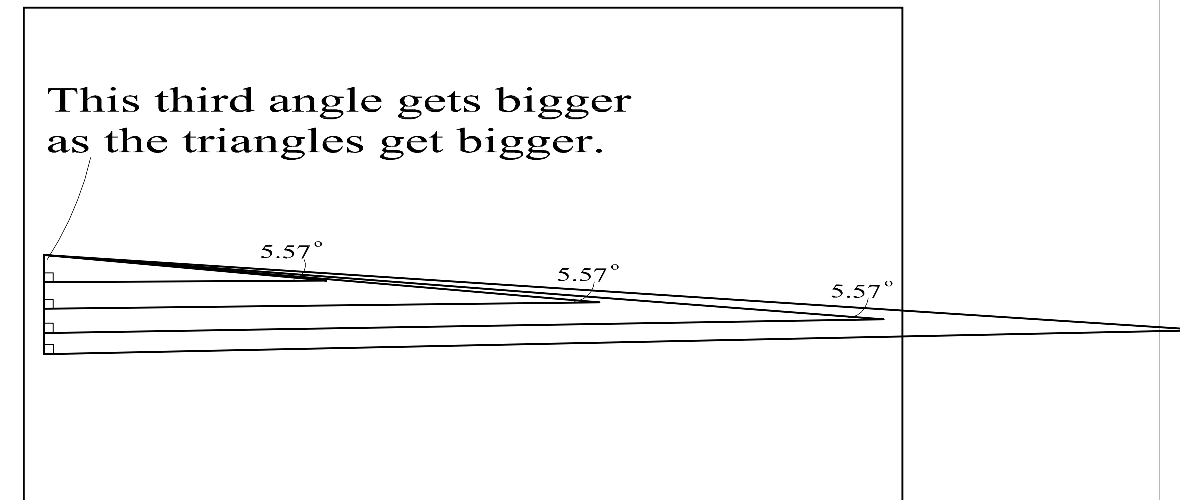
Type of Geometry
Hyperbolic
Constant factor
k = 50

If we copy the four different triangles of *Drawing 45*, and we re-arrange their positions, so that the corners with the angles that vary are stacked on top of each other, we get a different view of the triangles (seen here in *Drawing 46*).

As in Euclidean space, figures in Non-Euclidean spaces are "rigid". They can be moved around without changing their properties. We can rotate them, and mirror them, without changing the lengths of their sides or the angles measured at their corners; but we can not change their scale. They are not SIMILAR with respect to size.

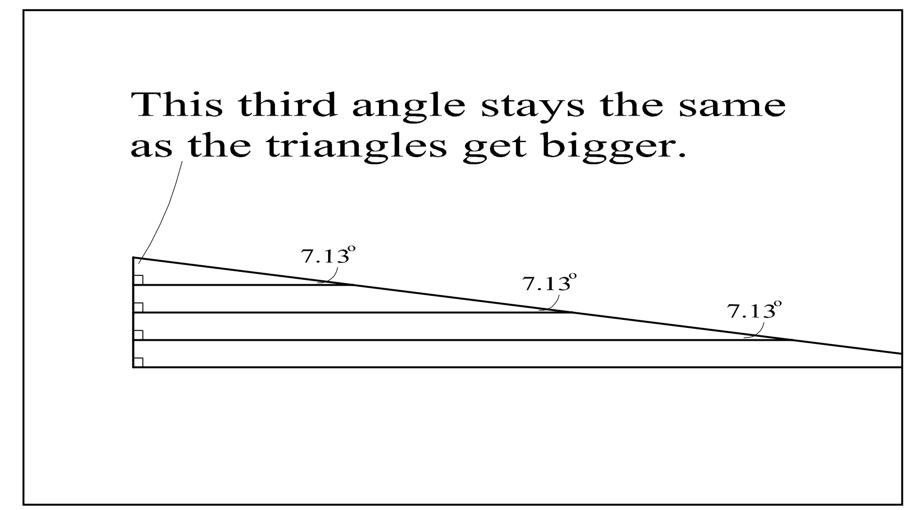
In Hyperbolic space, the bigger a triangles gets, the smaller the sum of its three angles becomes. In Elliptic space, the bigger a triangles gets, the larger the sum of its three angles becomes (up to a limit).

DRAWING 46: **The triangles of *Drawing 45* are arranged with the corner of the varying angle all stacked at the same point.**

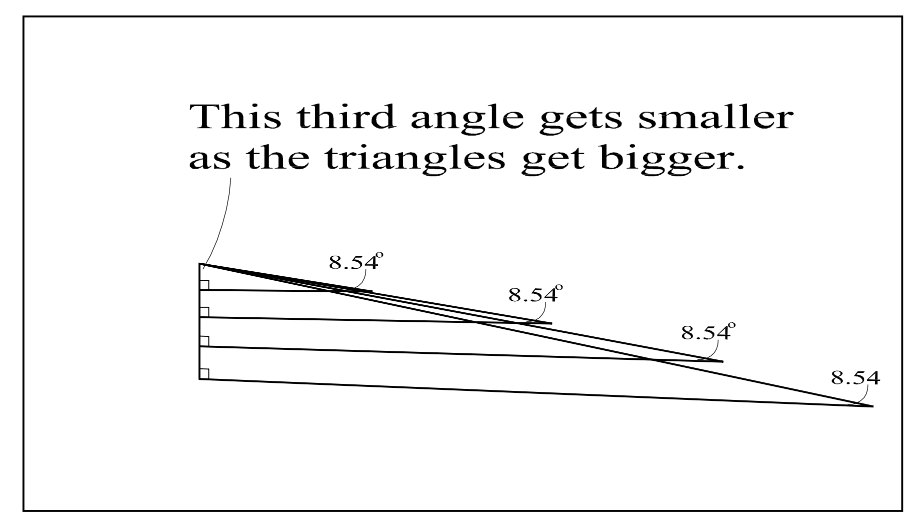


Type of Geometry
Elliptic

Constant factor
k = 50



Type of Geometry
Euclidean



Type of Geometry
Hyperbolic

Constant factor
k = 50

If we set up a segment of straight line within the view of our Perspective camera, we will see that it will always look smaller in Hyperbolic space, and bigger in Elliptic space.

The lines are set up in all three geometries in precisely the same manner, at the same distance from the Eye (the aperture of the camera), turned at the same angle.

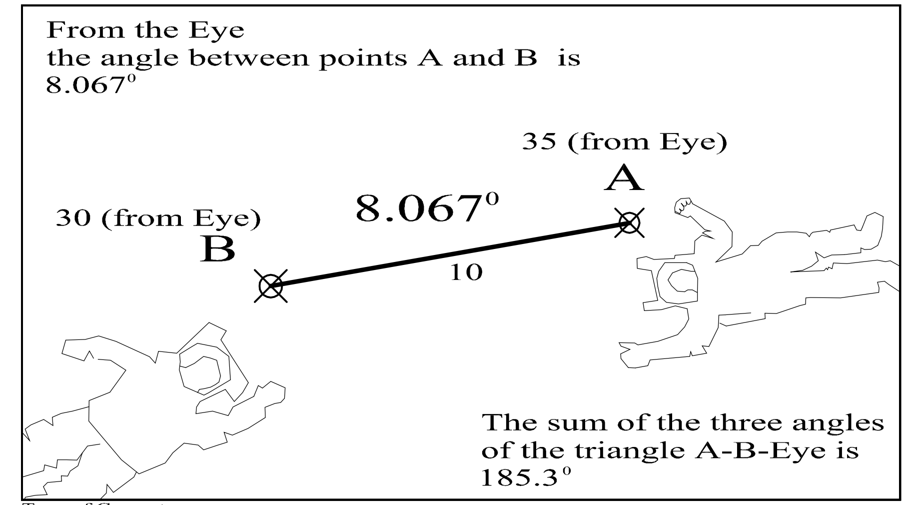
In Hyperbolic space, the line will look smaller than it does in Euclidean space. The angle seen from our Eye to the end points of the line becomes smaller. Just as our Eye sees the angles of any triangle getting smaller, when we measure this angle at its point of intersection, it will be measured as being smaller.

In Elliptic spaces, at short distances, short line segements will always look larger than in Euclidean space. Elliptic space in larger scales gets more complicated because every line is finite in length. A line starting from a point and moving outward in a straight line will eventually reach the point where it started.

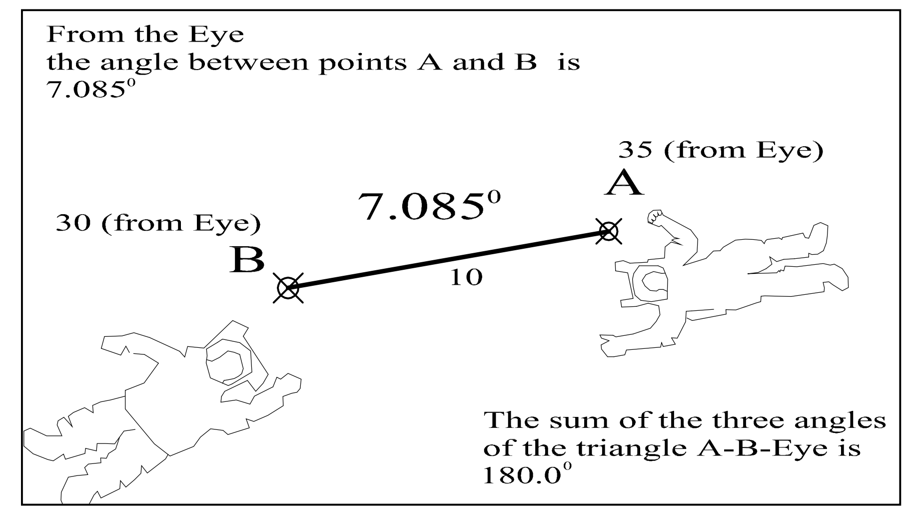
The perpendiculars we frst drew in *Drawing 41* will eventually intersect, then continue beyond. Whether or not they intersect once or twice depends on whether the Elliptic Geometry is assumed to be the "one pole" or the "two pole" version. This book will only discuss the two pole version of Elliptic Geometries. As a crude model one can imagine straight lines being like meridians of Longitude used to map the Earth, traveling perpendicular to the Equator and crossing at two poles. In this crude model the Eye would be the center of the Earth. In Perspective, straight lines of Elliptic Geometries always circuit completely around the Eye.

Regardless of how any line is positioned in space, a straight line will always appear straight, in every Perspective view.

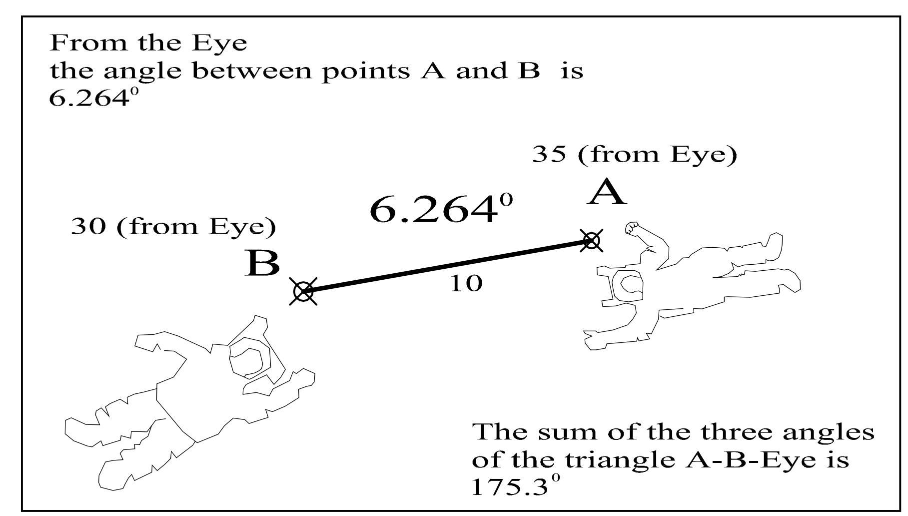
DRAWING 47: **The angle from the Eye to the end-points of any given line will vary in different types of geometry.**



Type of Geometry
Elliptic
Constant factor
k = 40



Type of Geometry
Euclidean



Type of Geometry
Hyperbolic
Constant factor
k = 40

Mathematically, the "k" factor is a radius. In the literature of Non-Euclidean Geometry I can find no special name for it. We might call it the "Radius of Density" for each space. In the past I've called it the "Radius of Curvature", but truthfully nothing is curving, so that name is a bit misleading.

We observe two surveyors, in *Drawing 47*, to illustrate the optical appearance of the varying density-of-distance in the spaces of our three geometries.

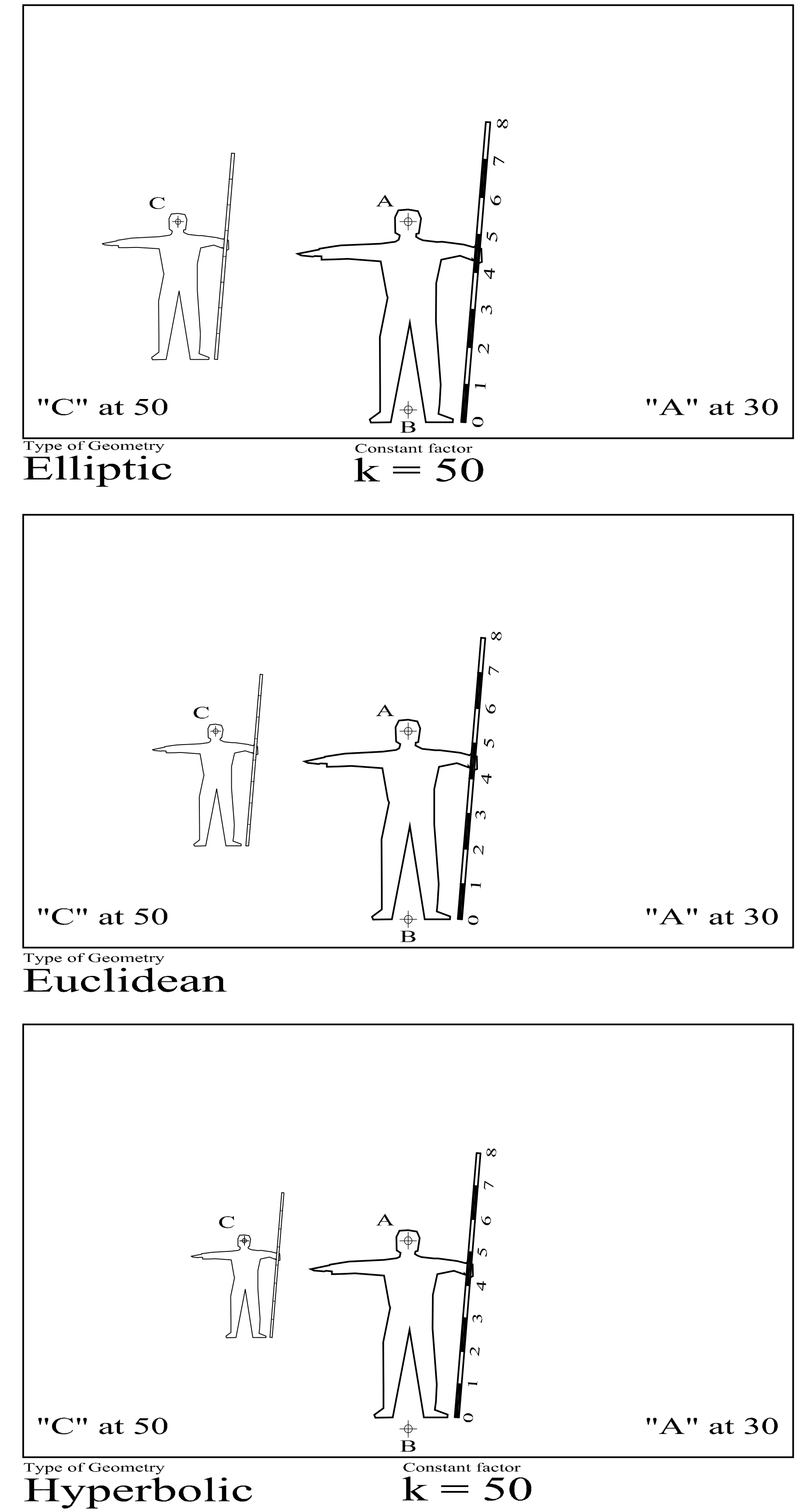
The points "A" and "B" are set to mark the same visual angle in all three illustrations. One can see that as space becomes "less dense" in the Elliptic Geometry, the figure appears bigger -- in Hyperbolic Geometry it appears smaller. The figures are all the same height, positioned at the same distance, 30 units, from the Eye to the figures (at point A).

The secondary figures are all set up at the same measured distance, 50 units, from the Eye of the Perspective observer to the eyes of the figures (at point "C") and a distance of 9 units perpendicular distance out from the line formed by the central ray of visual (the point in the center of the picture, passing through point "A").

In Elliptic geometry the image of the secondary figure gets smaller at a slower rate than in Euclidean space. In Hyperbolic the secondary figures gets smaller at a faster rate than in the Euclidean view. The farther into the distance we look, we faster the "density of length" changes.

While the lengths seem to be expanding or compressing objects, the meaning of length never varies. For the surveyors looking back at us, their local space seems "normal", their sizes "normal", while the objects they see in their distant views appear similarly expanded, or compressed, by the geometry of their space.

DRAWING 48: **Showing the relative densities of space. "A" and "B" are set at the same angle in all three views.)**

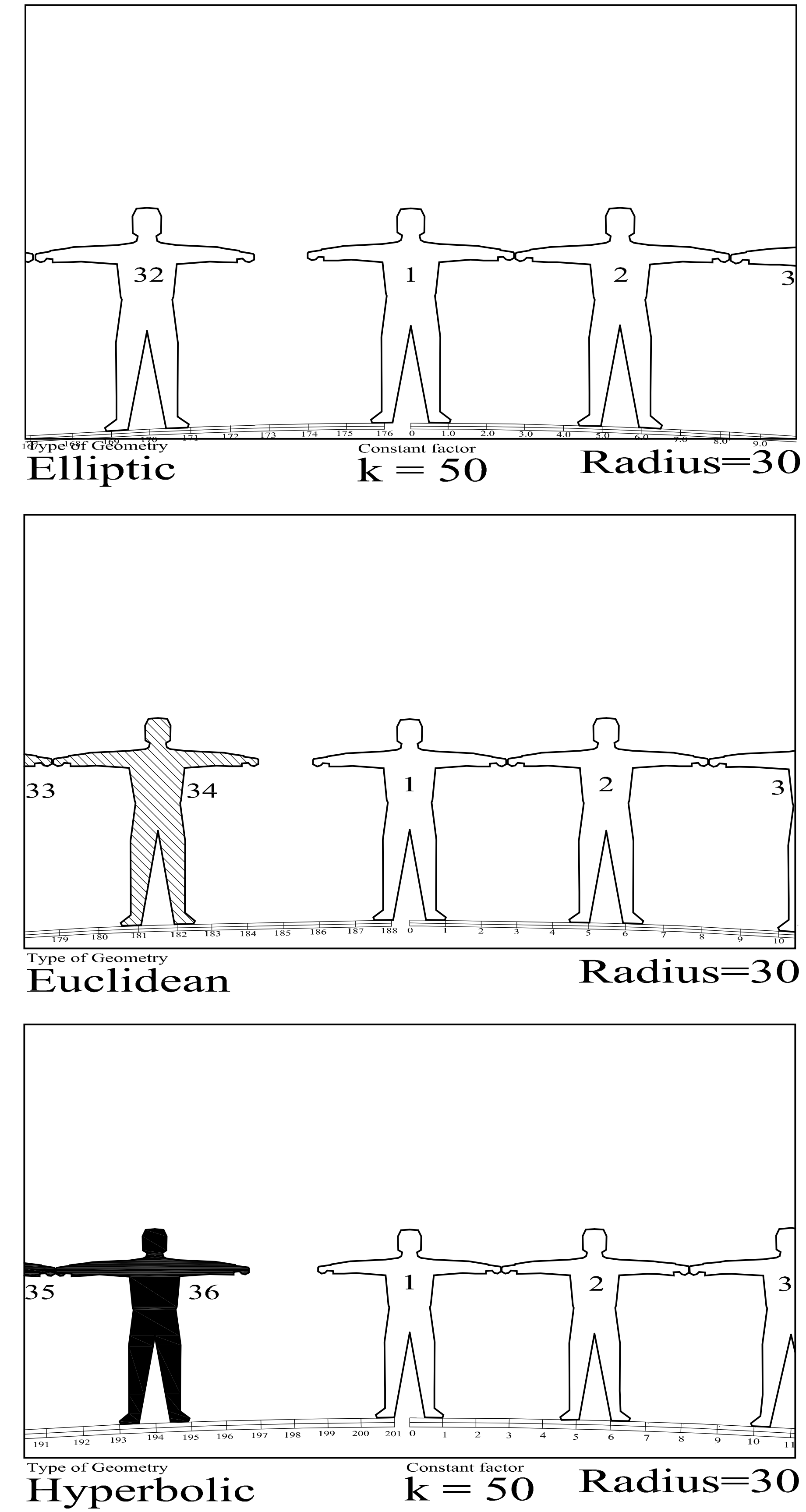


The change to the density of space is not merely an optical effect, there really is "less space" (less distance) or "more space" (more distance) enclosed within the unchanged angles.

Here, in *Drawing 49*, we line up equally sized figures to form a circle around the Eye of our Perspective observer. A measuring tape is set at their feet.

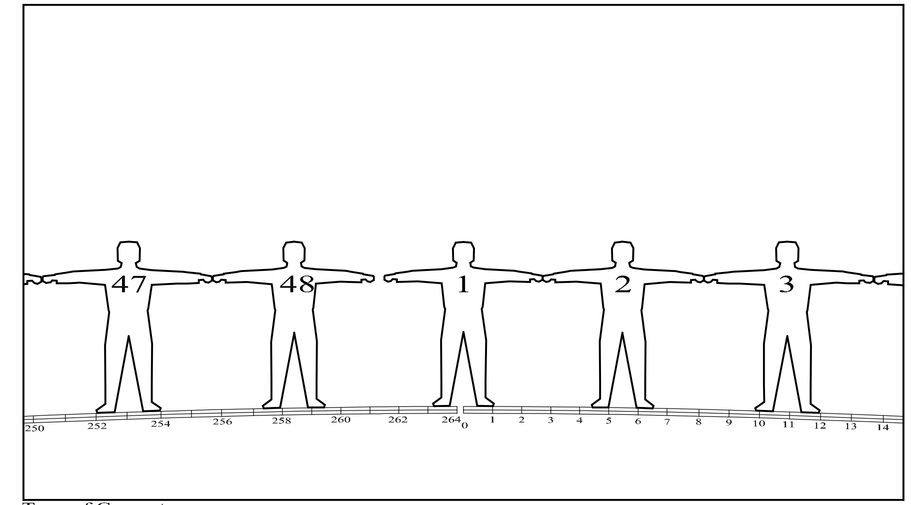
In Elliptic Geometry, at a radius of 30 units of length, there are fewer figures in our circle, its circumference is less. In Hyperbolic Geometry, at the same radius of 30, there are more figures needed to complete the circle -- its circumference is longer.

***DRAWING 49:* The measured length of the circumference of a circle changes in Non-Euclidean geometries.**



As the radius of the circle is increased, the variation to the length of its circumference is amplified. The effect becomes greater with greater distance.

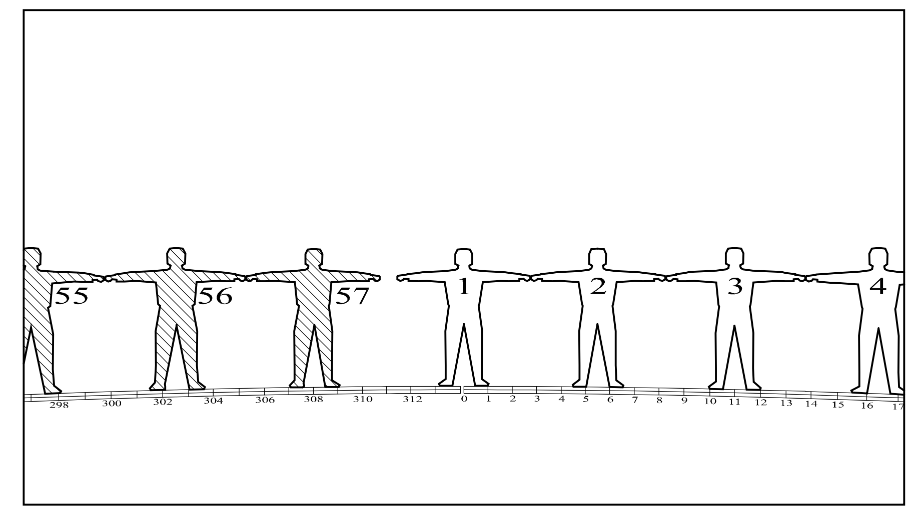
DRAWING 50: **The larger the radius of the circle is set, the greater the proportion of change to its circumference.**



Type of Geometry
Elliptic

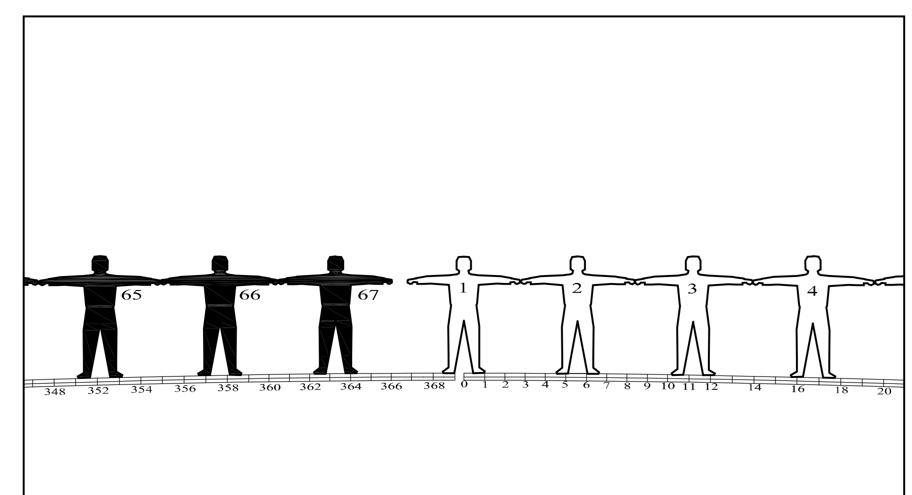
Constant factor
k = 50

Radius=50



Type of Geometry
Euclidean

Radius=50



Type of Geometry
Hyperbolic

Constant factor
k = 50

Radius=50

If we position the Eye of our Perspective observer above, and look down at a series of concentric circles, *Drawing 51* illustrates what will be seen.

(The order of the geometries is reversed on this page, to let the image of Elliptic circles extend past the arbitrary fixed edge of the picture plane.)

It can readily be seen by the growing proportion of fully rendered figures standing in the circles, that the length of the circumferences are increasing in the Hyperbolic geometry, and decreasing in the Elliptic space.

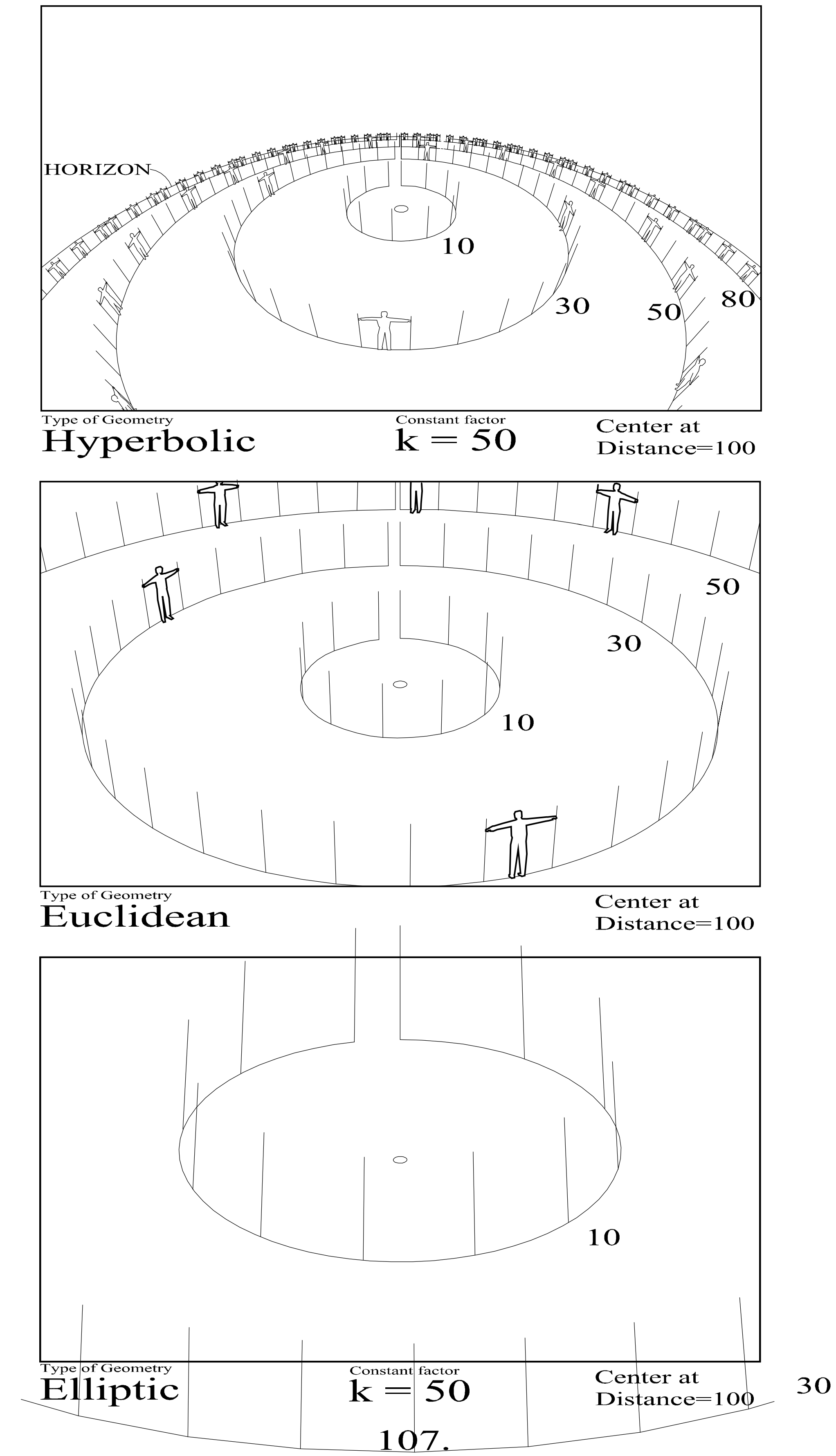
Circumference of a Circle

Radius of Circle	k=50	Hyperbolic	Euclidean	Elliptic
	10.0	63.25 (11 Figures)	62.83 (11 Figures)	62.41 (11 Figures)
	30.0	200.01 (36 Figures)	188.50 (34 Figures)	177.39 (32 Figures)
	50.0	369.20 (67 Figures)	314.16 (57 Figures)	264.36 (48 Figures)
	80.0	746.31 (135 Figures)	502.65 (91 Figures)	314.03 (57 Figures)
	120.0	1,717.27 (312 Figures)	753.98 (137 Figures)	212.20 (38 Figures)

Seen from a distance, the Non-Euclidean planes start to appear warped, or curved. This is an optical effect, the plane remains totally flat, in the same sense that it is flat in Euclidean space.

In the chart of Circumferences it should be noted that after reaching a certain distance, the circumference of circles in Elliptic spaces start to get smaller, until finally the radius travels half the finite length of a line and the circumference becomes a single point. Paradoxically, though the plane appears to encircle the Eye of the obserever, it never really curves, but remains perfectly flat.

DRAWING 51: **Concentric circles, seen from above. Lines show places, while only the additional figures are fully drawn.**



To construct this *Drawing 52*, we imagine a flat plane at 5 units distance below the Eye, like an infinite flat floor upon which the Perspective observer stands. At a distance of 60 units straight out from the Eye, a perpendicular line is set upon the flat plane. Standing along this straight line is a line of figures, each exactly 5 units apart, with their eyes at 5 units height.

Every possible straight line drawn on the plane will appear as a straight line on the Perspective picture plane.

Points at the top of the head of each figure are connected to form a continuous line. In Euclidean space it is a straight line; in Elliptic space it curves upward; and in Hyperbolic space it curves downward. These are indeed real curves, called "Equidistant Curves". No two straight lines in either of our basic Non-Euclean Geometries can travel along at the same distant apart (what we call "Parallel" in Euclidan geometry). The points at an equal distance to any straight line forms an Equidistant Curve.

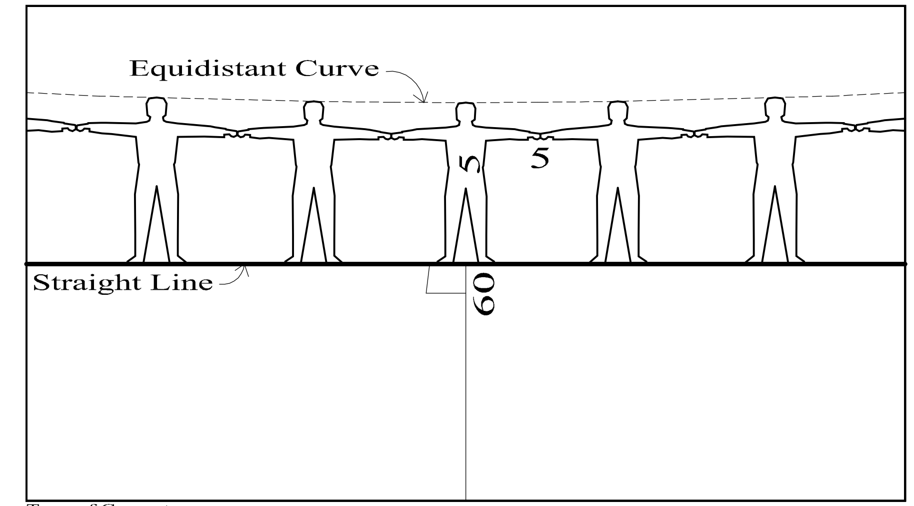
Horizons are formed at infinite distants in Euclidean and Hyperbolic geometry. In Euclidean space the Horizon of the flat plane appears as a straight line behind the eyes of the standing row of figures. Despite the plane being flat, in Hyperbolic space the Horizon always appears as a curve, and that curve is one of the important figures in analytical Hyperbolic Geometry. It has been given various names: the Boundary Curve, Limiting Curve, the Absolute, Horocycle, Oricycle.

In Elliptic Geometry there is no Horizon, flat planes appear to encircle the Eye.

Finally the figures appear to turn. In the Euclidean space they will all appear to face forward with parallel gazes, but in Non-Euclidean spaces the figures appear to turn. This is not purely an optical effect, though the figures remain standing on the straight line looking outward in a perpendicular direction.

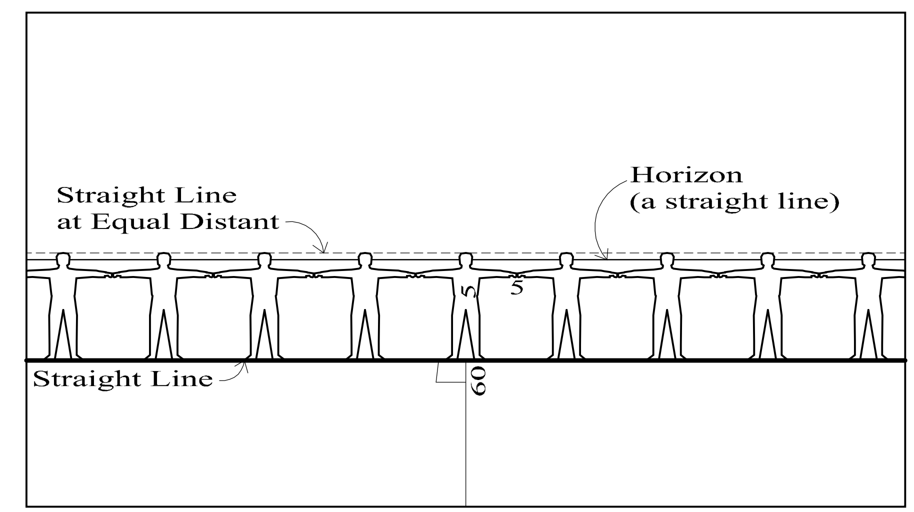
In Elliptic geometry the figures appear to turn inward, toward the observer; in Hyperbolic space as the line proceeds toward the edge of the picture plane, the figures appear to turn outward.

DRAWING 52: **Figures standing on a straight line on a plane, their eyes at the same height as the Perspective observer's.**

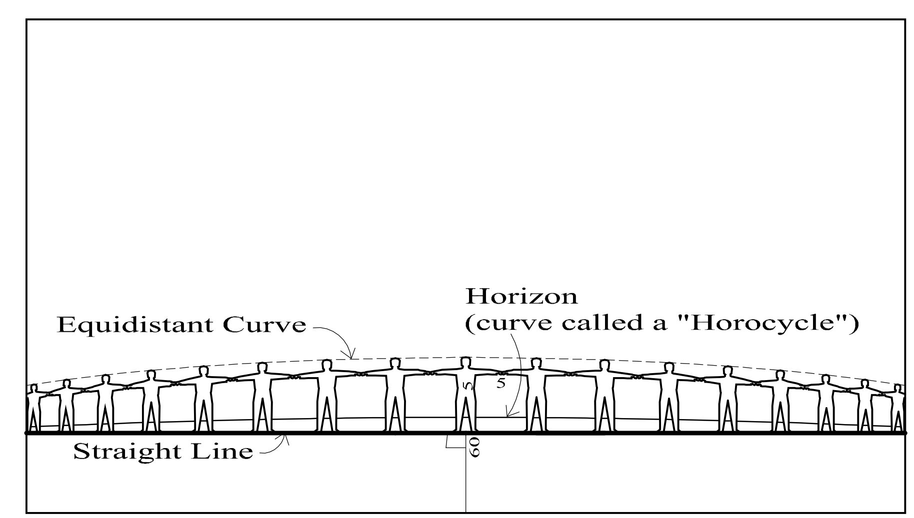


Type of Geometry
Elliptic

Constant factor
k = 40



Type of Geometry
Euclidean



Type of Geometry
Hyperbolic

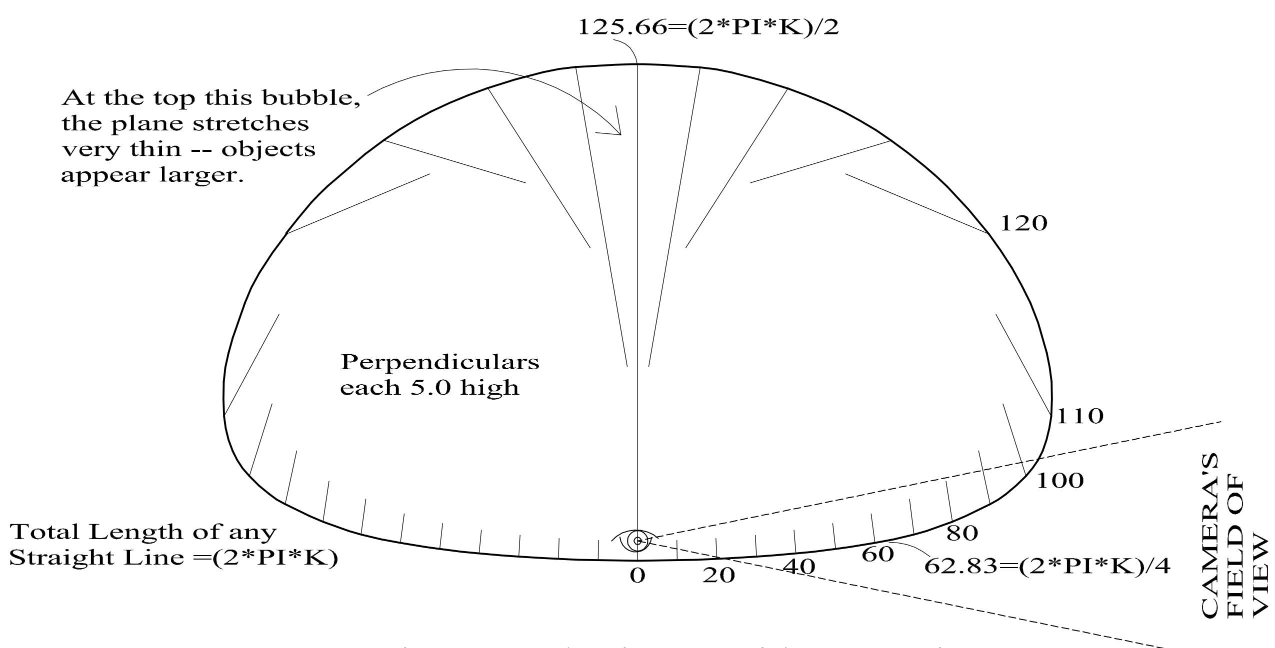
Constant factor
k = 40

In *Drawing 52* the figures are standing at a fixed distance from a straight line.

The curving line upon which the feet of the figures are standing in the Non-Euclidean spaces is an Equidistant Curve, and the line connecting the tops of their heads is also an Equidistant Curve. (This incidentally illustrates how any line equal distance from an Equidistant Curve also forms another Equidistant Curve.)

All lines in Elliptic geometries have finite lengths. If you proceed far enough along a straight line in Elliptic space you return to your point of origin. Optically, in Perspective views of Elliptic spaces, all straight lines and flat planes appear to wrap around the Eye of the Observer.

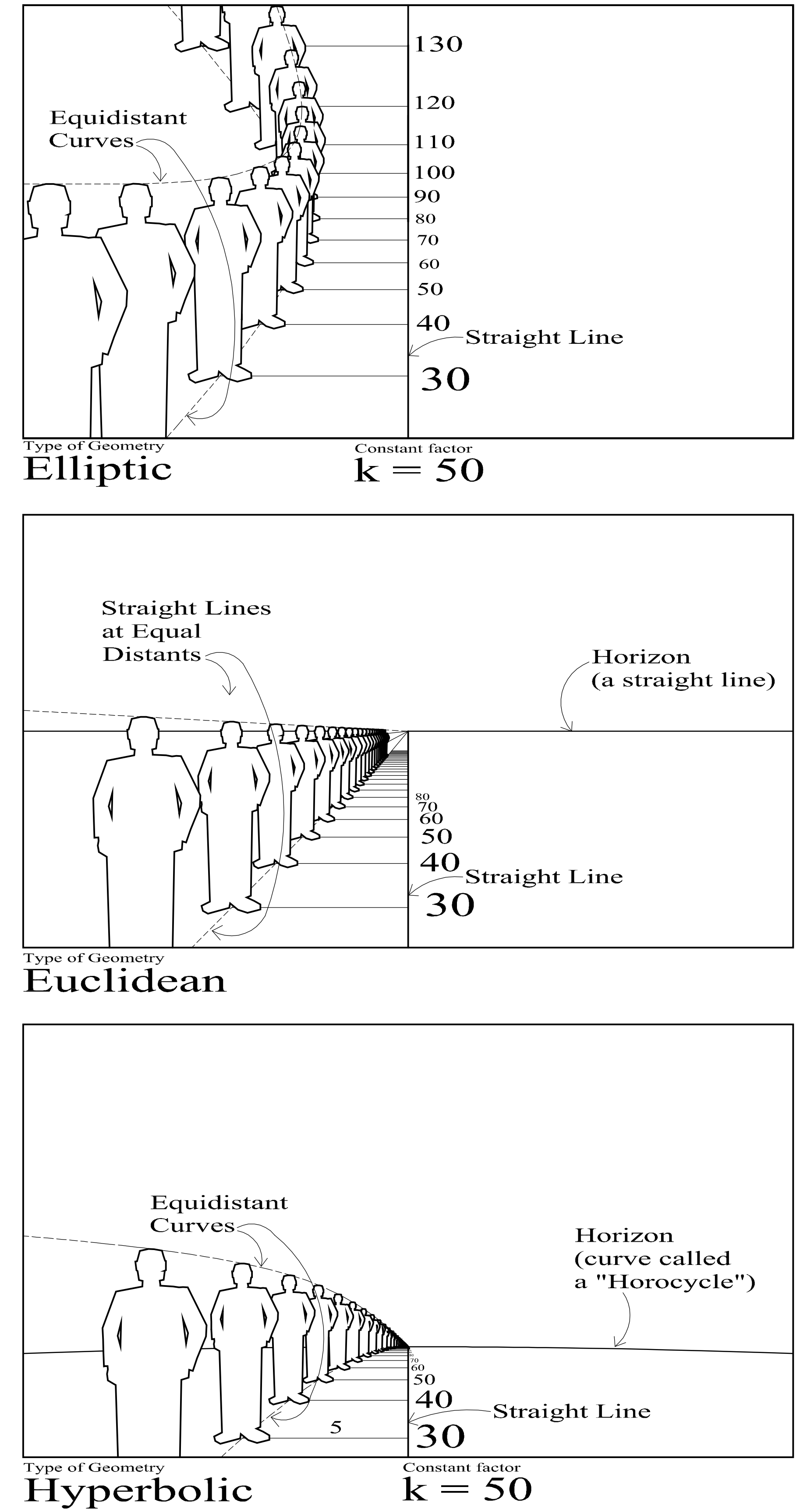
A plane wrapping around the Eye can appear as a precise sphere when that plane is at one certain distance from the observer, but for all the other cases, the sphere deforms like an elastic bubble. We may draw a scaled Euclidean diagram of the distance of a "section" cut through the bubble-like surrounding plane, and it looks like this:



"Distance-Angle Diagram" of the Perspective view of a flat Elliptic plane (k=40) with perpendicular line segments standing upon it.

The Eye and its picture-window may rotate at its fixed position without altering the bubble (or perpendiculars).

DRAWING 53: **Figures of equal height standing at a fixed distance from a central straight line.**



A SHORT DIVERSION ABOUT "HORIZONS"

In the Perspective of Euclidean Geometry an observer at any elevation above a flat ground plane will see the Horizon as a straight line at Eye level (and we say that this straight line is "horizontal"). In theory (if there was no atmospheric dust or limit to visual acuity) we could see the entire distance of the surface, stretching infinitely outward.

Young artists are taught to draw this Horizon as a straight line at Eye level.

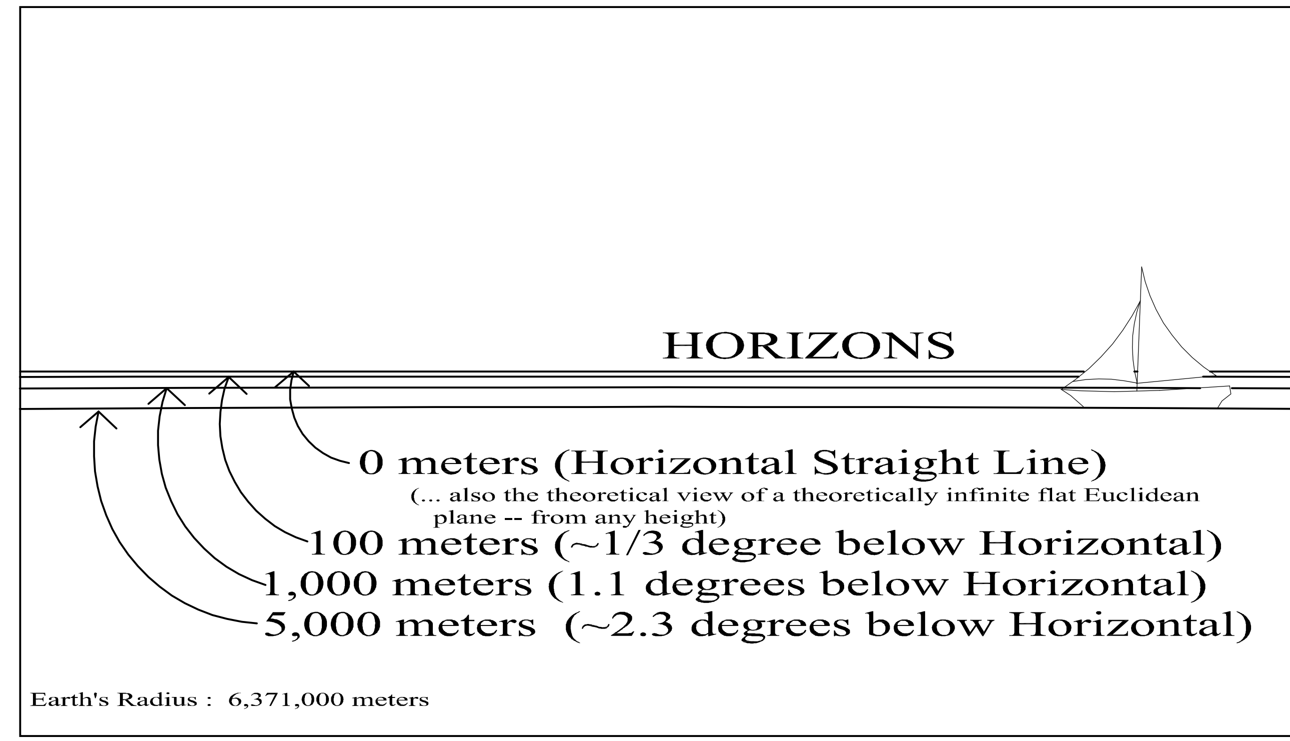
In fact though, when you look across a large body of water you often are not be able to see the opposite shore. The range of your view is limited by the curvature of the Earth.

The curvature of the Earth cuts off views of distant ojects suprisingly quickly. If you stand at a high enough elevation, at least in theory you can see the curvature of the Earth's surface. In fact this usually gets obscured by atmospheric dust or mists.

[Also, there is an optical effect of visual curvature when straight lines are viewed over wide angles. Perspective is only deemed "realistic" only within a cone of vision never exceeding sixty degrees, but human eyes typically are seeing a field of view three times wider. If you stand fairly close to a long flat interior wall and look up at the straight line along the intersection of the flat wall and a flat ceiling, you tend to see that straight line as a curve.]

A Hyperbolic plane's curving Horizon reminds me of the curving Horizon of our round Earth, but we are seeing the entire Hyperbolic surface, every point along the line of our vision, extending out to infinity. The Hyperbolic plane is not curving, there is no part of its surface curling down, out of our line of sight.

DRAWING 54: **Perspective appearance of Horizons -- for a flat Euclidean plane and a spherical Earth seen from various heights.**



Euclidean (or "k" being very large)

Eye level at 2 meters elevation:
3.1 miles of visibility to Horizon

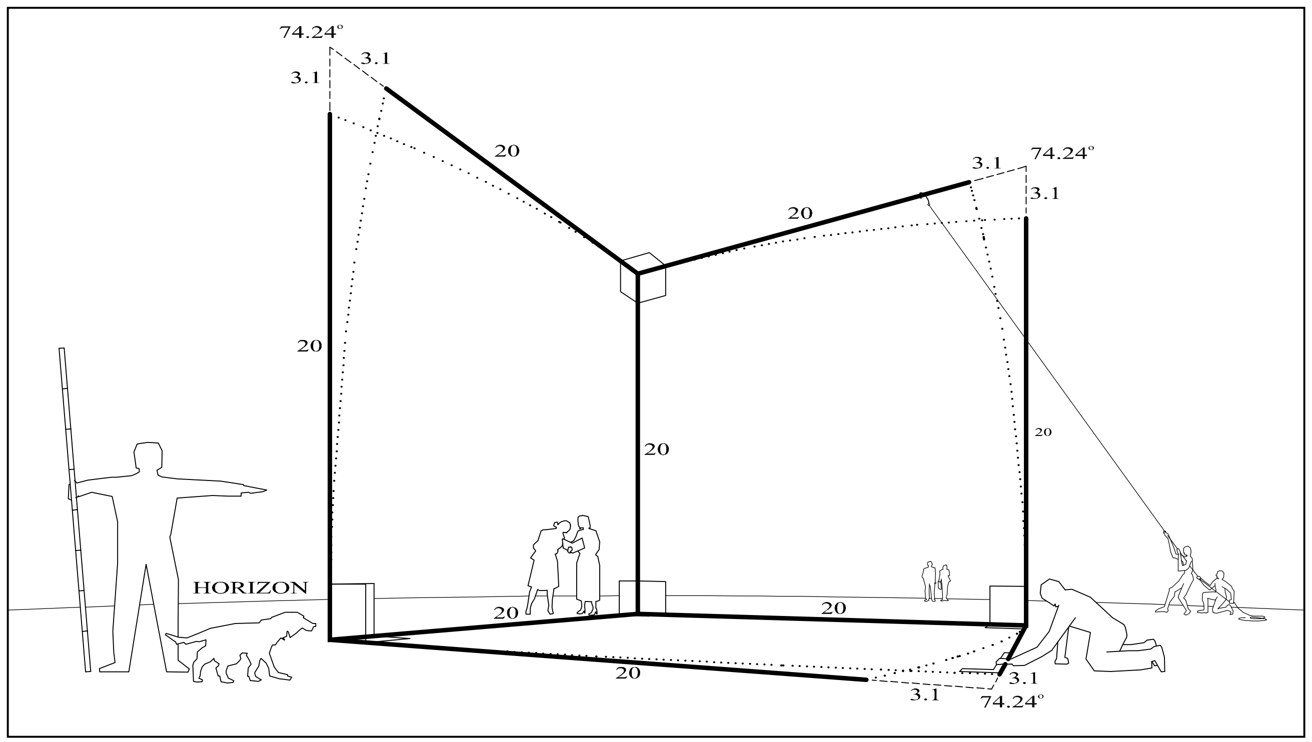
Eye level at 100 meters elevation:
22.2 miles of visibility to Horizon

Eye level at 1,000 meters elevation:
70.1 miles of visibility to Horizon

Eye level at 5,000 meters elevation:
157.0 miles of visibility to Horizon

DRAWING 55: **Three Lambert Quadrilaterals set as a corner: ARRANGEMENT I.**

Trying to set up networks to calculate the Perspective pictures of Non-Euclidean figures, I discovered that the corners of right angled figures could fit together in a variety of different ways. Here are separate illustrations of four basic arrangements, two shown in Hyperbolic Geometry and two in Elliptic Geometry.



Three equilateral "Lambert Quadrilaterals" Hyperbolic geometry: k=40
Arrangement I -- set perpendicular to each other (three interior right angles meeting)

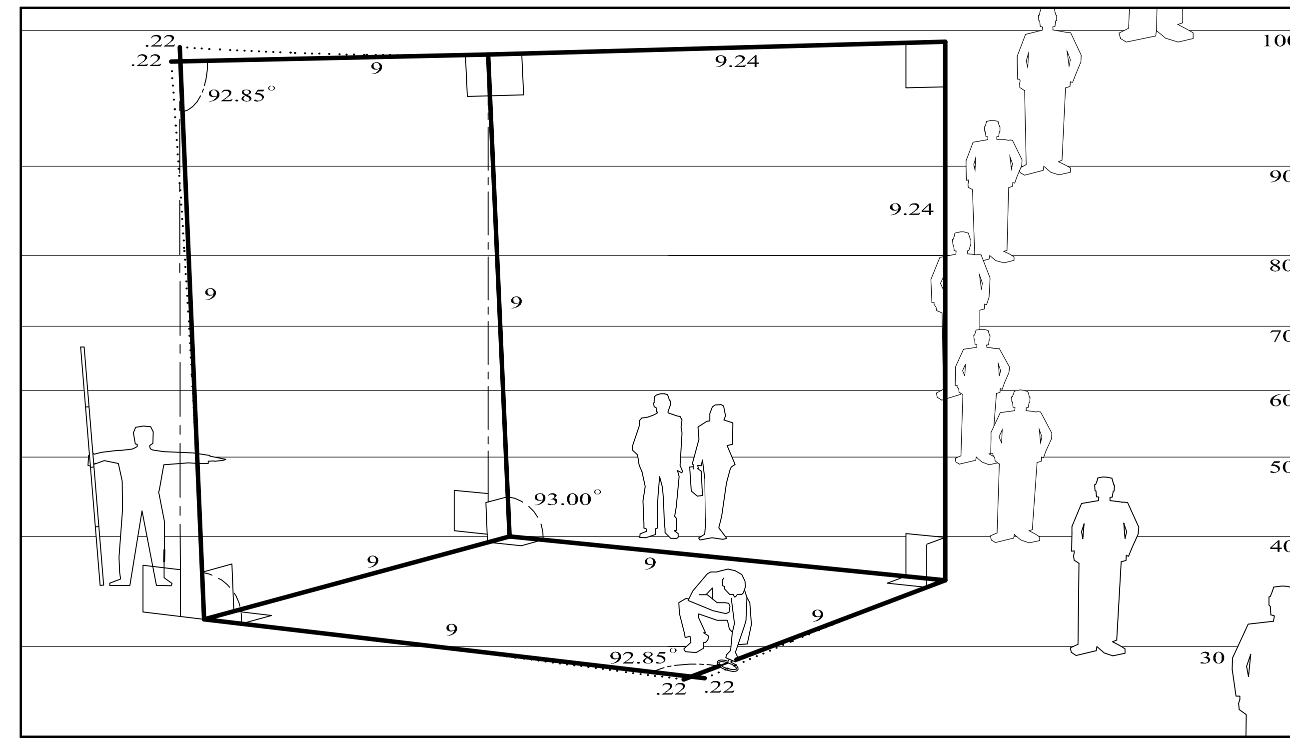
In this ARRANGEMENT I, three right-angled figures are set mutually perpendicular to each other, their faces at right angles.

One can readily see why a system of Cartesian coordinates can not be constructed in Non-Euclidean space,

DRAWING 56: **Three Lambert Quadrilaterals set as a corner: ARRANGEMENT II.**

ARRANGEMENT II is set up in Elliptic space.

The three right angles of the Lambert Quadrilatereal in the far right is altered so that one of its perpendicular corners is on the outside. The fourth corner, now at the junction point of the three planes, becomes an angle greater than 90 degrees, and to keep the edges of the three qualrilaterals joined, the planes start to rotate.



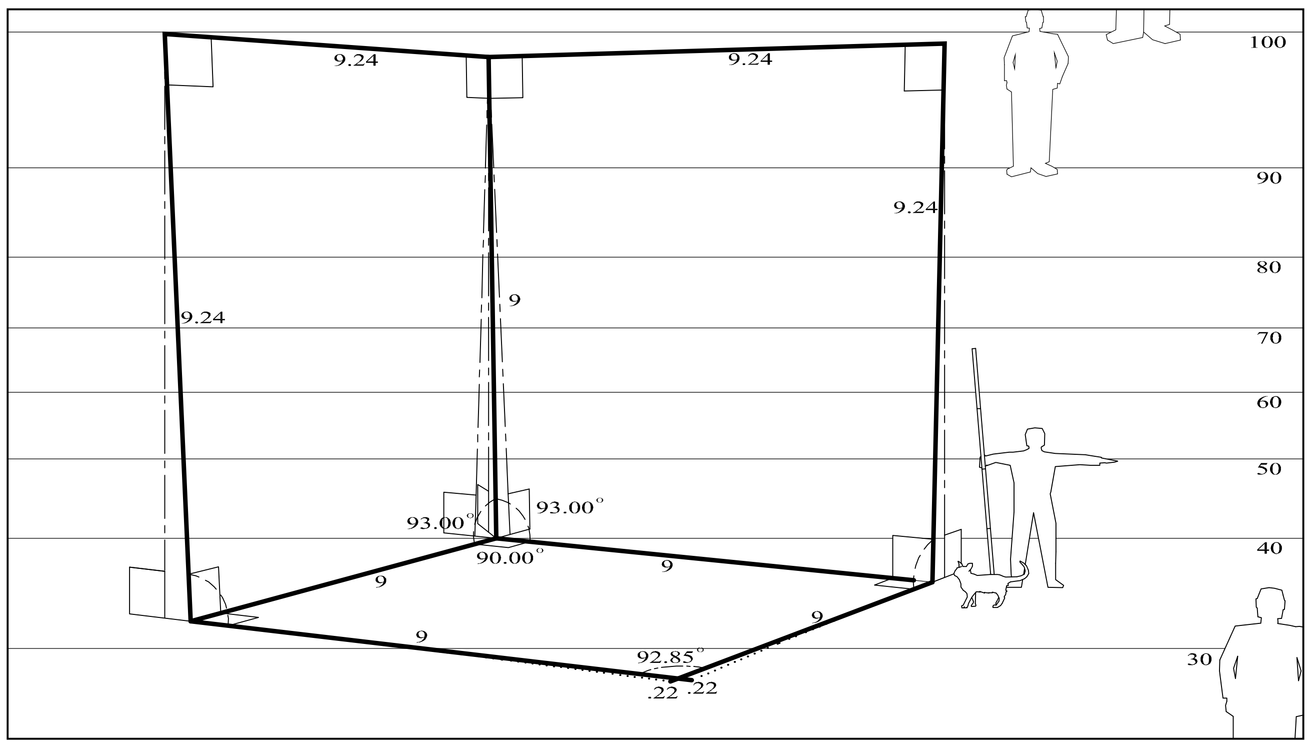
Three equilateral "Lambert Quadrilaterals" Elliptic geometry: k=40
Arrangement II -- Two interior right angles meeting, One exterior right angle

With the four-sided quadrilateral on the left now leaning outward, we drop down new lines perpendicular to the ground plane, in order to calculate the positions of its corners and edges.

The ground plane of Elliptic space encircles up and around the Eye of the Perspective observer.

DRAWING 57: Three Lambert Quadrilaterals set as a corner: ARRANGEMENT III.

In this third ARRANGEMENT, only the quadrilateral lying on the ground plane has a right angle in the interior corner, where the three planes intersect.

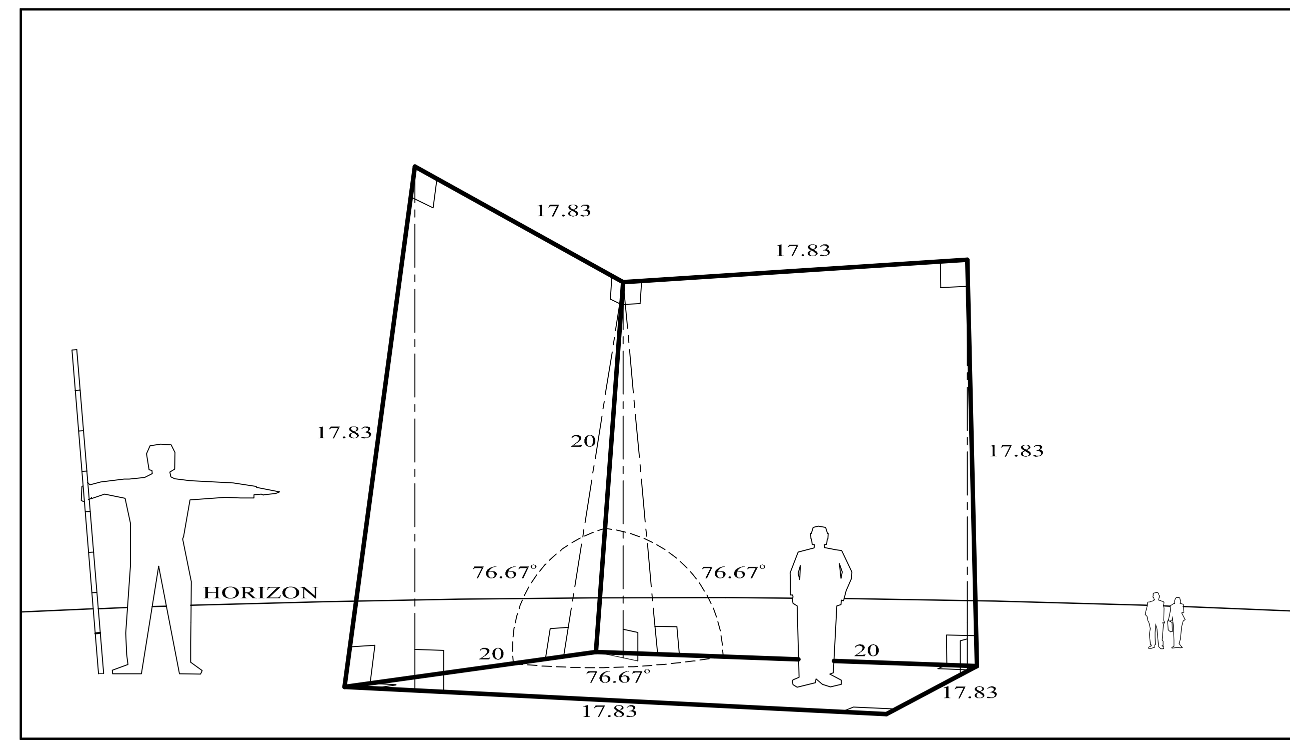


Three equilateral "Lambert Quadrilaterals" Elliptic geometry: k=40
Arrangement III -- One interior angle. Two exterior right angles.

The ground plane of Elliptic space encircles up and around the Eye of the Perspective observer.

DRAWING 58: Three Lambert Quadrilaterals set as a corner: ARRANGEMENT IV.

ARRANGMENT IV is set in Hyperbolic space. Opposite to Elliptic geometry, now the fourth angle of the quadrilateral is always less than 90 degrees. In this arrangment, all the corners at the interior intersection are non-perpendicular, so the figures all lean inward, toward one another.

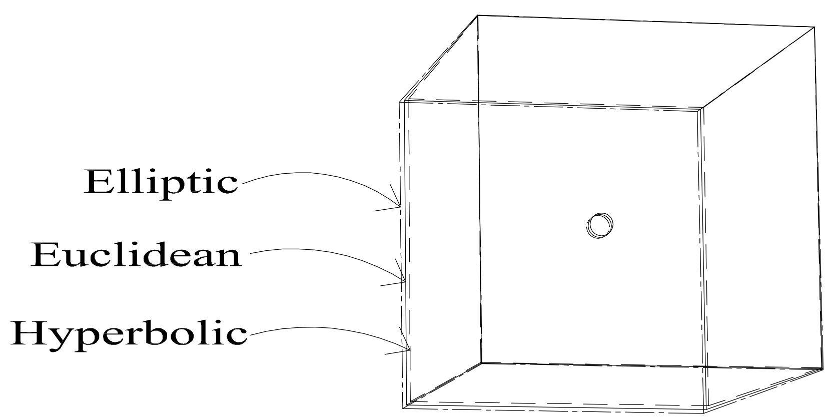


Three equilateral "Lambert Quadrilaterals" Hyperbolic geometry: k=40
Arrangement IV -- All right angles on exterior corners

Non-Euclidean regular polyhedrons look very much like their Euclidean versions. It is both a bit surprising and perhaps a bit dissappointing.

I can think of no reason to believe there could be any additional Non-Euclidean "regular solids".

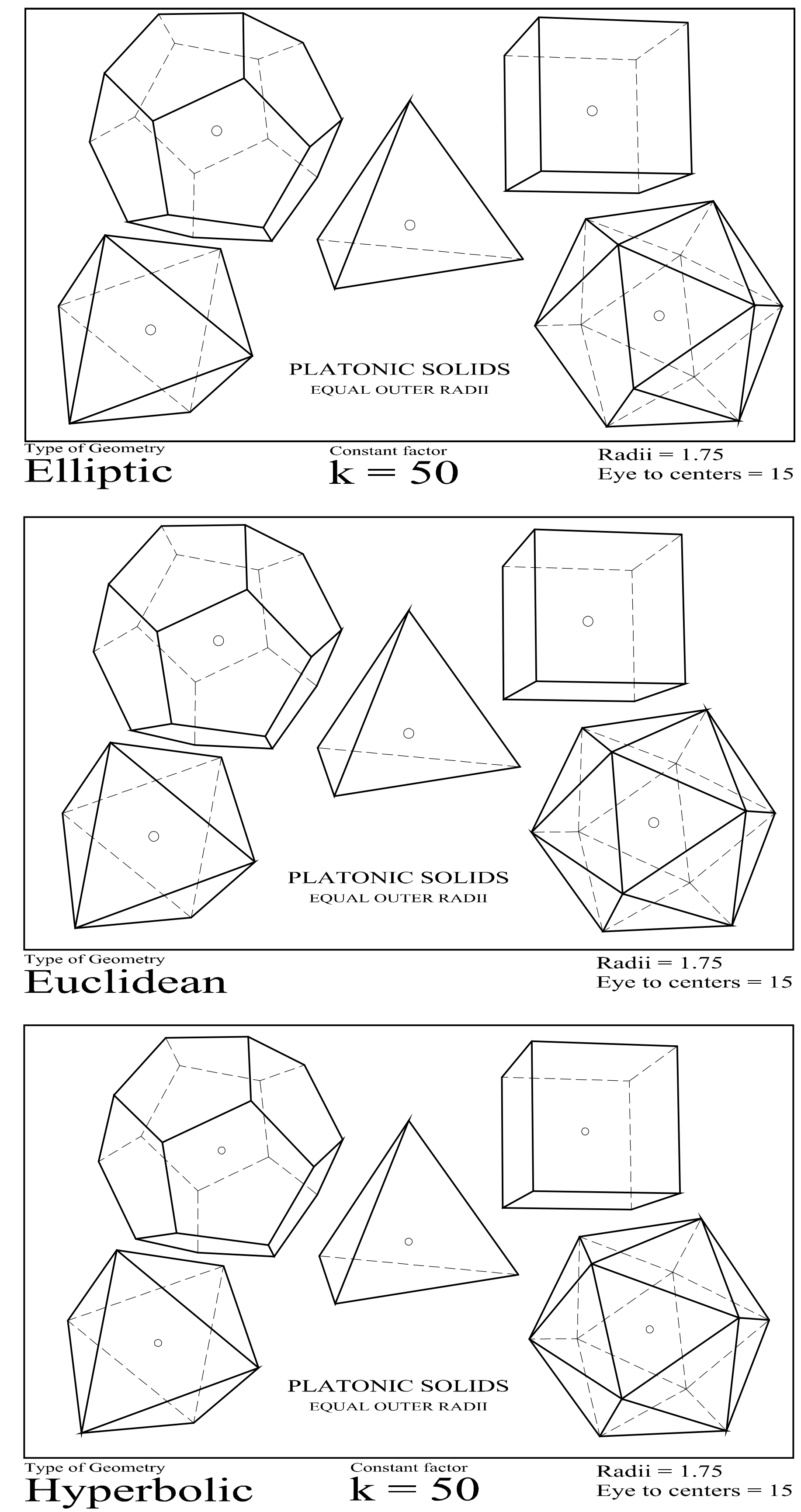
Non-Euclidean regular polyhedrons share much with the Euclidean versions. Each has straight lines of equal length for edges. Each has flat surface faces, matching in size and geometry (congruent to each other) all around the polyhedron.   A line from the center of polyhedron to the center of each face consistently forms a right angle; and a line from the midpoint of each edge to the center of the polyhderon also consistently forms a right angle.



The polyhedrons of the three different geometries tend to look identical (except for a slight difference in size), but actually they are not. There are slight differences in the "rate of foreshortening" of the three Perspective images (as seen in the Cube, above).

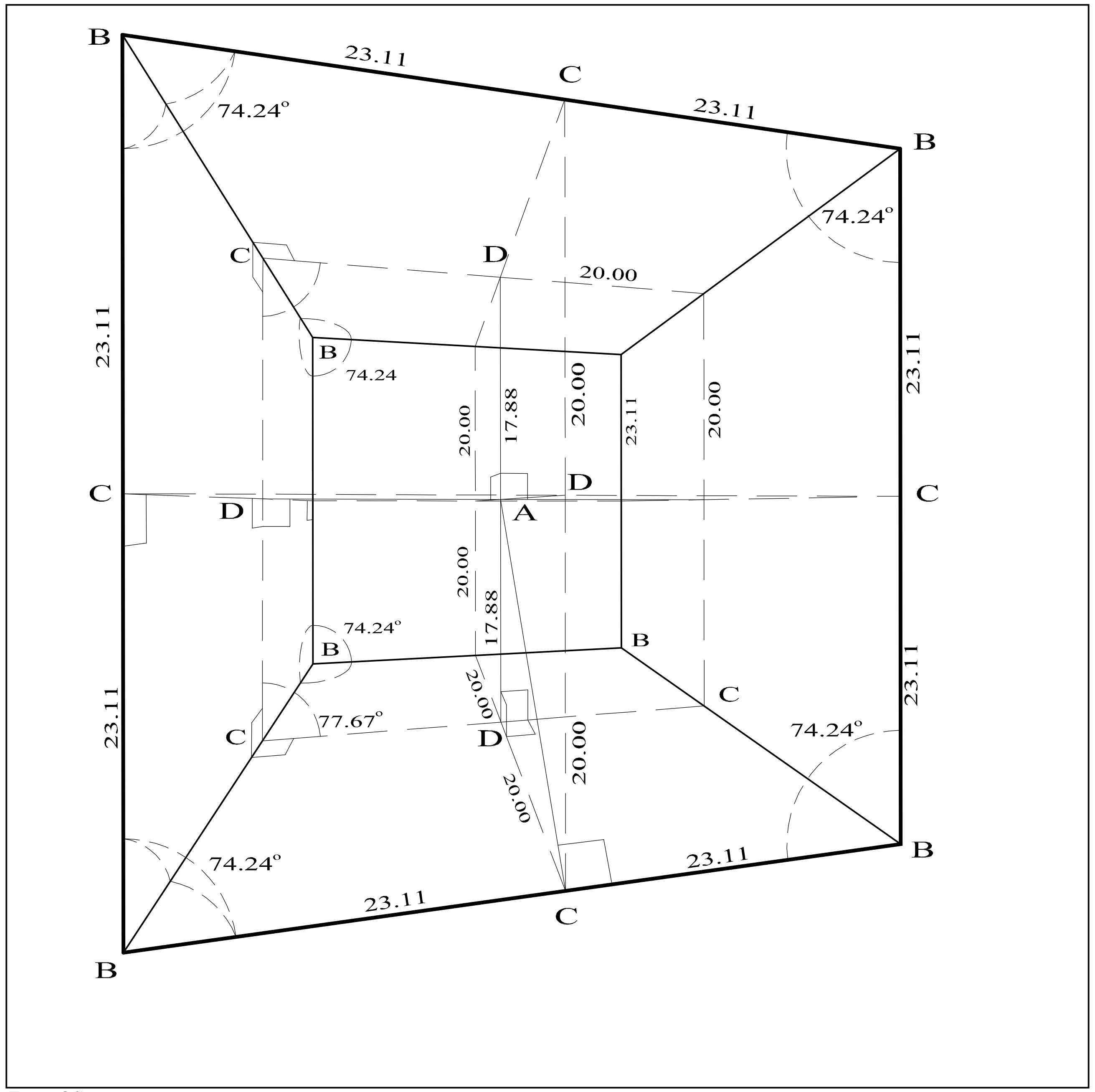
And when we examine the angles and dimensions of the Polyhedrons themselves, we see that each of the Non-Euclidean solids is very different from its Euclidean version.

DRAWING 59: **The five regular polyhedrons**



DRAWING 60: A six-sided regular polyhedron, a Cube, in Hyperbolic space.

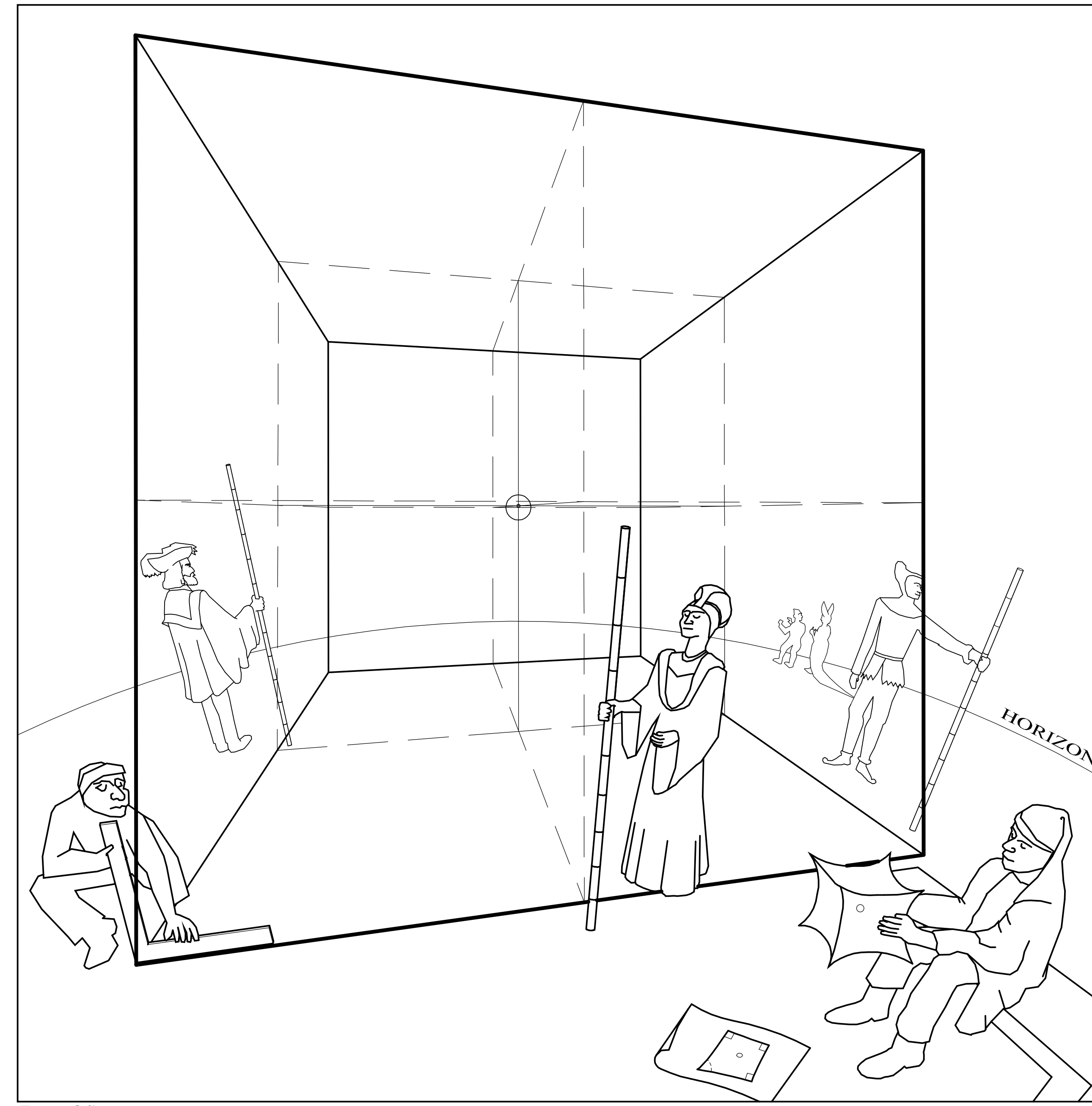
Two studies of the same cube -- one with measurements, and a second with a crew of surveyors holding measuing rods precisely perpendicular to the flat plane on which the flat bottom face of the cube stands.



Type of Geometry
Hyperbolic

Constant factor
k = 40

DRAWING 61: **The same Cube, with surveyors.**

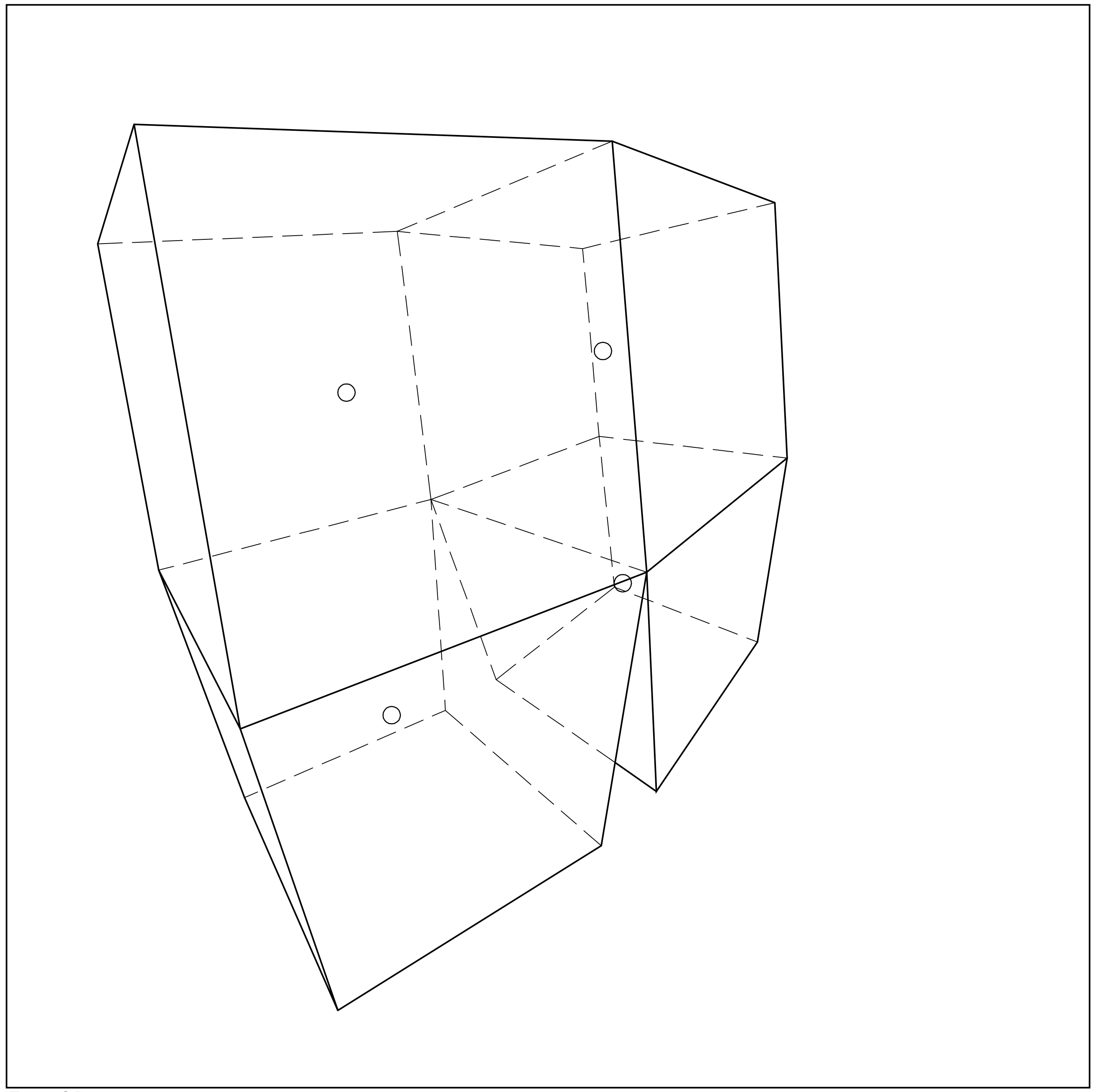


Type of Geometry
Hyperbolic

Constant factor
k = 40

DRAWING 62: **Four regular cubes set with faces adjoining.**

When we try to nest multiple regular Non-Euclidean polyhedrons together we discover how different the spaces of different geometries have made them.



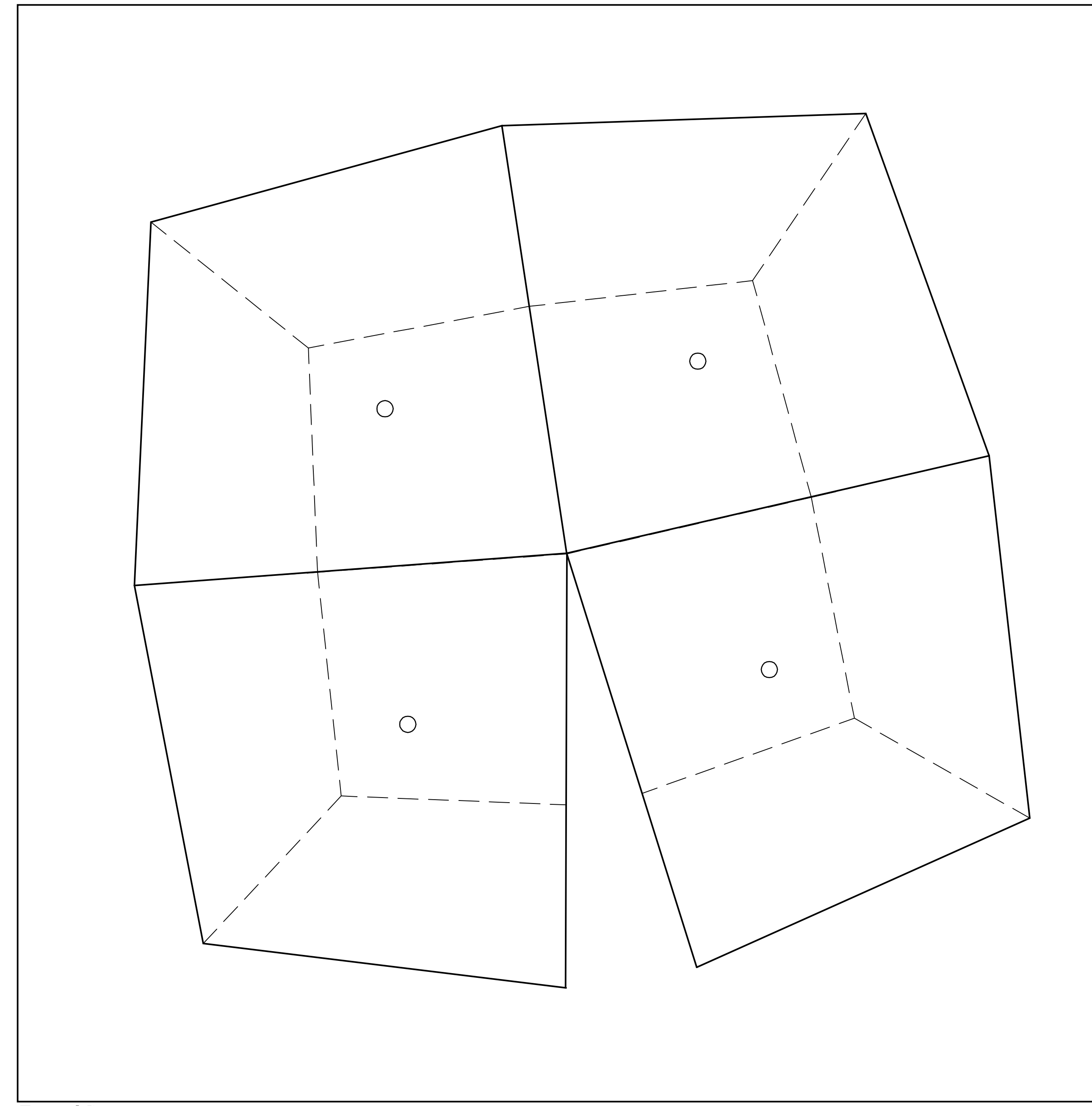
Type of Geometry
Hyperbolic

Constant factor
k = 40

There is a famous instance where one specific size of regular duodecahedrons form precisely perpendicular angles and can be packed in an infinite continuous array (similar to cubes in Euclidean space).

DRAWING 63: **The same cubes viewed from a different angle.**

Here, the Eye of the Perspective observer is set up to look directly down the single straight edge line common to all four cubes.



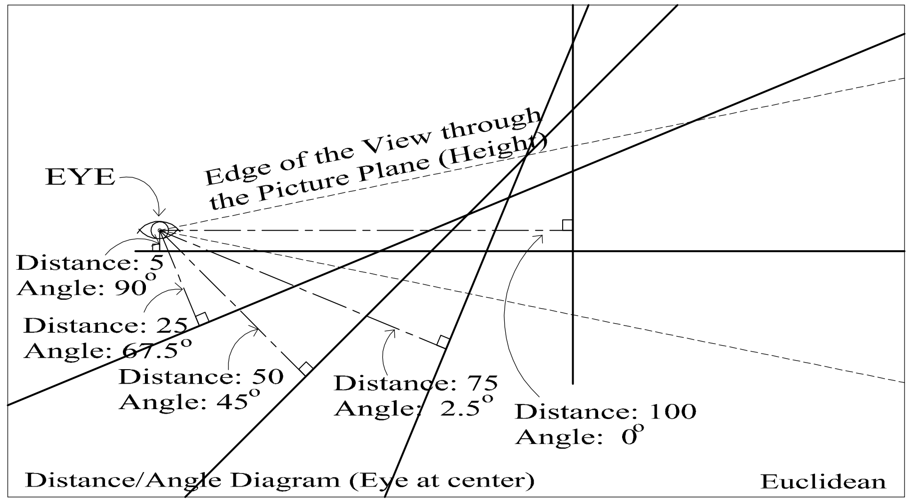
Type of Geometry
Hyperbolic

Constant factor
k = 40

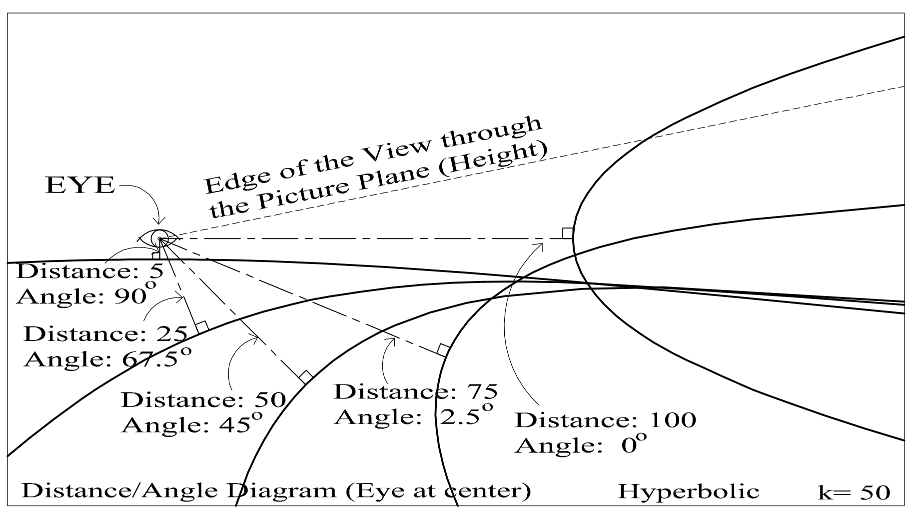
In all geometries a Perspective view precisely down the end of straight line segment is a point; and the view precisely along the edge of a flat plane is a straight line.

The Perspective Eye views a flat plane marked with a series of concentric circles starting from a center point defined by the line drawn from the Eye meeting the plane at precisely a right angle. Along the circles are perpendicular uprights of various heights.

For the Euclidean plane, a "distance-diagram" showing the various positions of the plane looks like this:



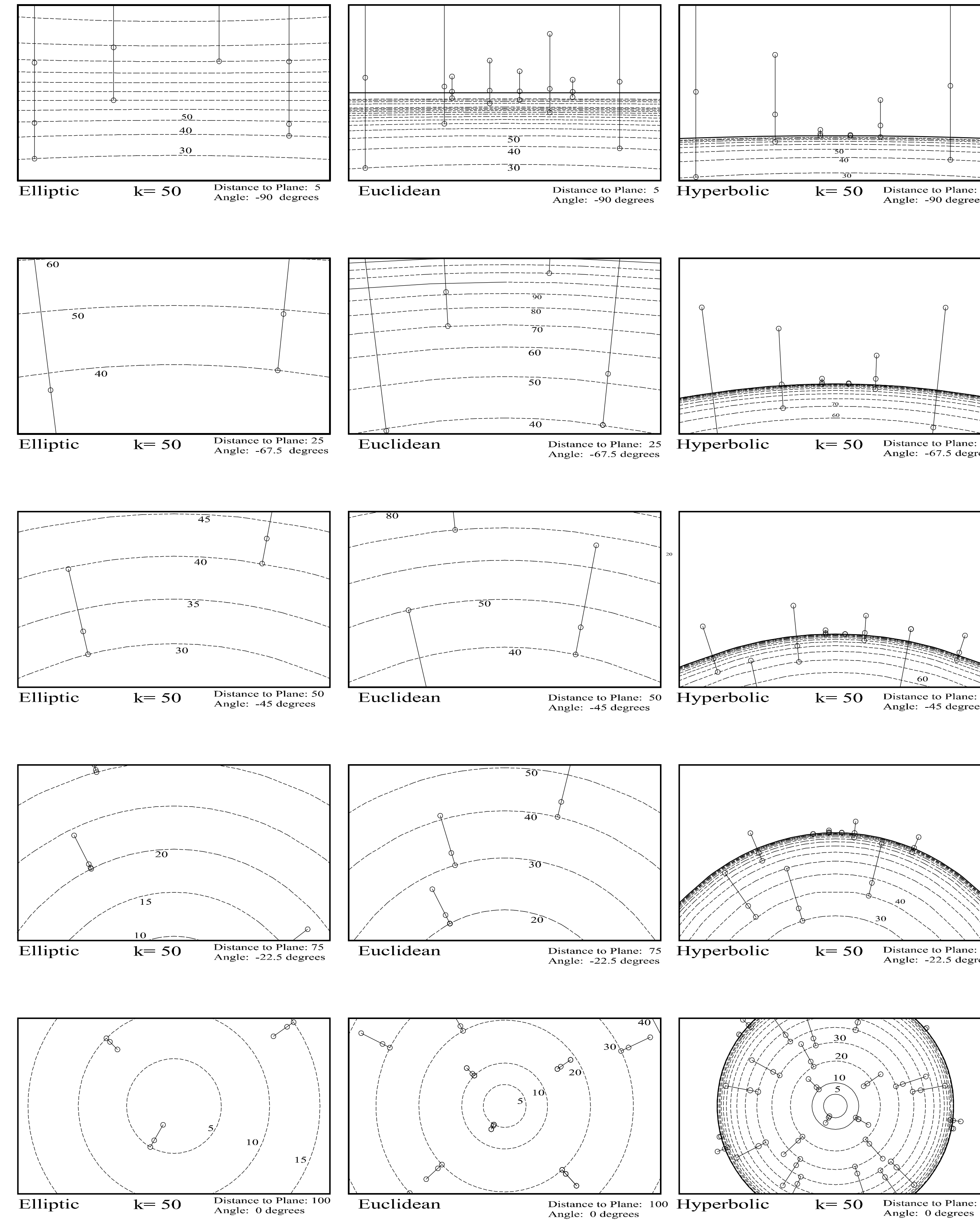
The planes are actually flat in Hyperbolic space, but when diagrammed in Euclidean space (according to the distances from the Eye at various angles) the planes appear as curved surfaces.



For the Elliptic plane the "distance-diagram" looks similar to the elastic bubbles seen in *Drawing 40* (page 84), the Eye rotating to the various angles used here.

In all three Geometries, the "vanishing point" for lines perpendicular to a plane is the point on the Perspective picture-plane (inside or outside the picture frame) made by the projection of the solitary perpendicular from the given plane passing through the Eye.

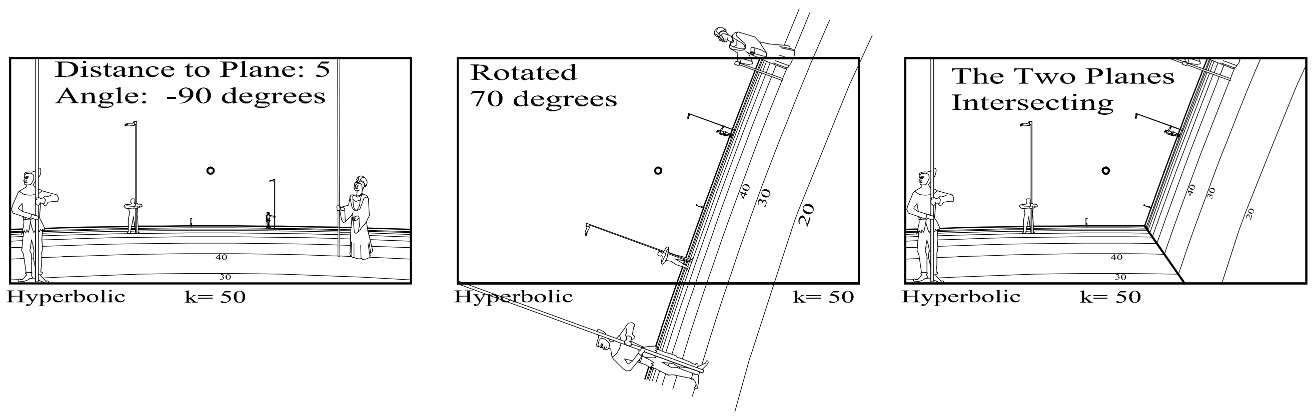
DRAWING 64: **Perpendicular line segments of various lengths standing on a flat plane, viewed from various heights and angles**



Where two planes intersect the points along their intersection form a straight line. This is true in Hyerbolic and Elliptic Geometries as well as Euclidean.

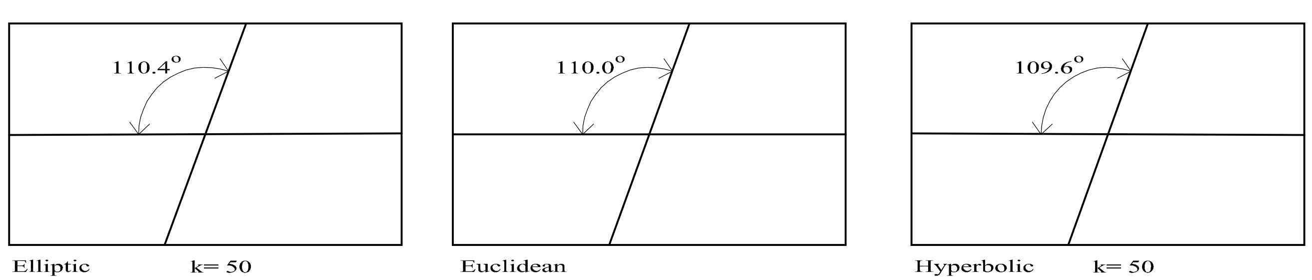
Perspective images may be spun around the central light of sight (the line perpendicular to the picture-plane passing through the Eye). If we rotate the construction of the object viewed, its image will not change except to rotate around the center of our picture plane.

For example we take our first Hyperbolic plane, and it's rotatation with respect to the Central Ray of Vision is a simple rotation of the picture image.



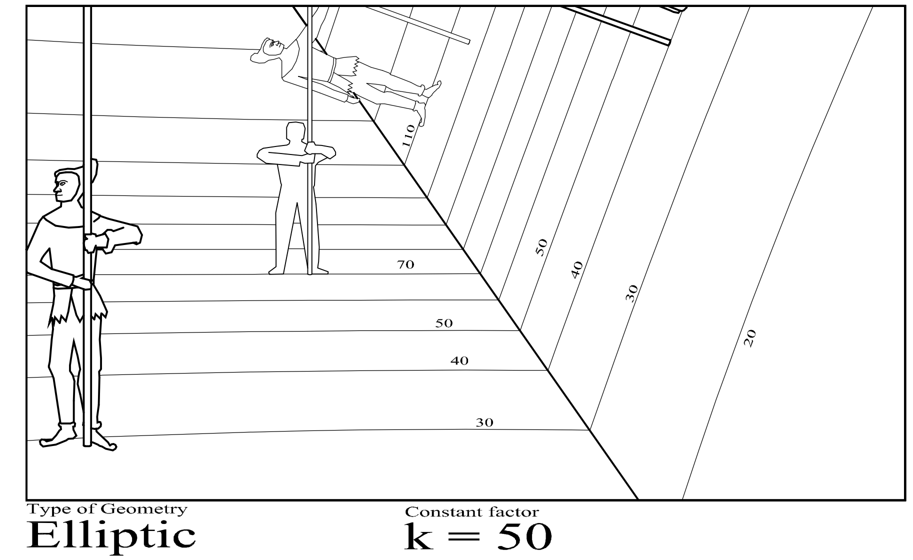
Combining the two planes together, their intersection forms a straight line.

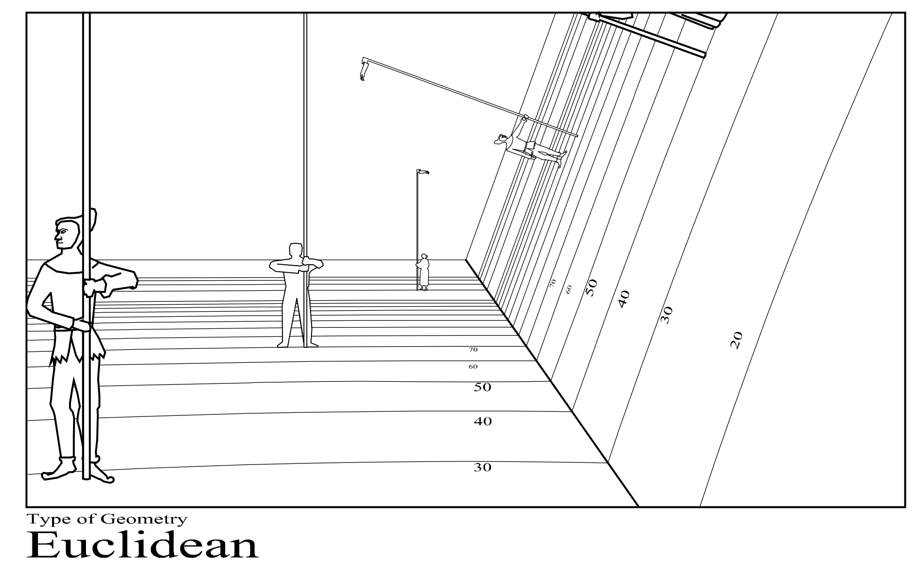
If we relocated the two intersecting planes so that our Eye looked exactly down the straight line of their intersection we would see the intersecting line as a single point, and the plane as straight lines. (In Elliptic Geometry the plane technically encircles the Eye, on both sides, but what small portion is not within the line of the plane is so far away, and stretched so thin, that it is practically invisible.)

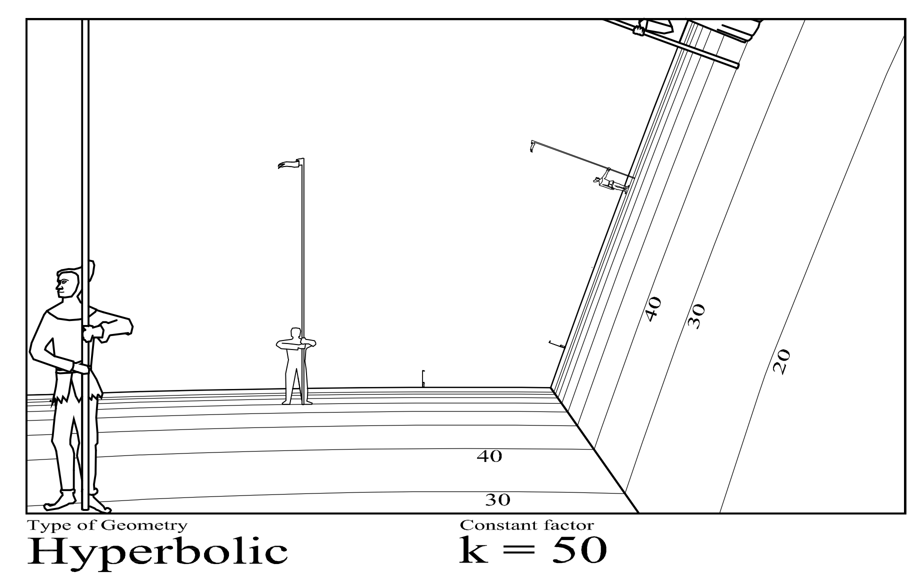


Because the planes have been set up according to an angle made by the perpendicular-to-the -Eye, their intersection angles vary slightly (between the three geometries).

DRAWING 65: **Two planes intersecting from a straight line and a uniform angle between the planes all along that line.**





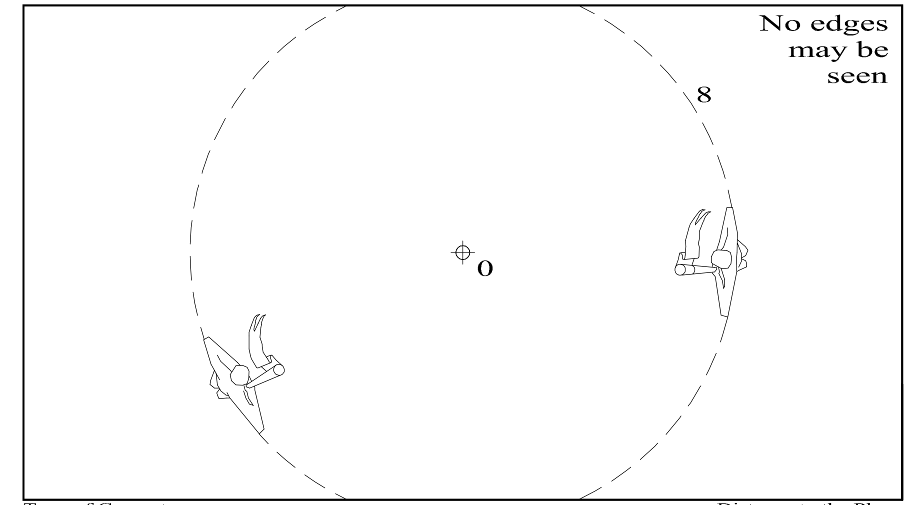


This Chapter's final series of drawings will analyze, in detail, the Perspective apearance of flat planes viewed in a straight-forward perpendicular manner.

In any of our three geometries, equilateral triangles can fit together into close-packed "tilings". In Euclidean space six identical equilateral triangles can fit together at one corner. In Non-Euclidean spaces similar tilings of equilateral triangles are possible only at a certain side length for a certain value of "k". In Elliptic Geometry five equilateral triaangles with a side length of 22 work for a value of "k", almost equal to the "k" value in Hyperbolic geometry where seven equilateral triangles of the same 22 unit side length fit together in close-packed "tiling".

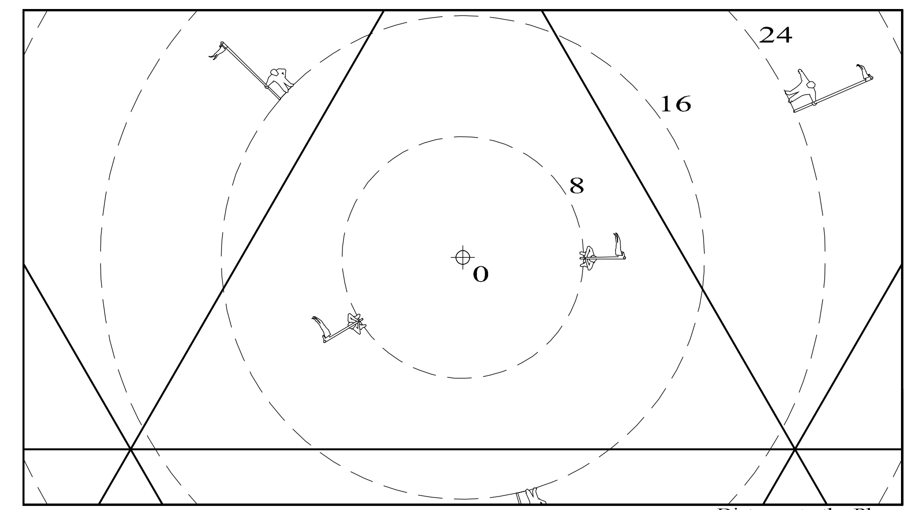
Viewed at the same distance as the other two geometries, the five-equilateral-triangle-tiling of Elliptic space is so big that we can not see an edge, and even in Euclidean space a single triangle is too big to fit entirely within the view of our camera's Eye. But in Hyperbolic space this same setup shows the full width of the infinite plane, with 7-triangles meeting in corners dissappearing quickly into the distance.

***DRAWING 66:* "Tiling" of equalateral triangles fitted together on a flat plane viewed in a perpendicular manner.**



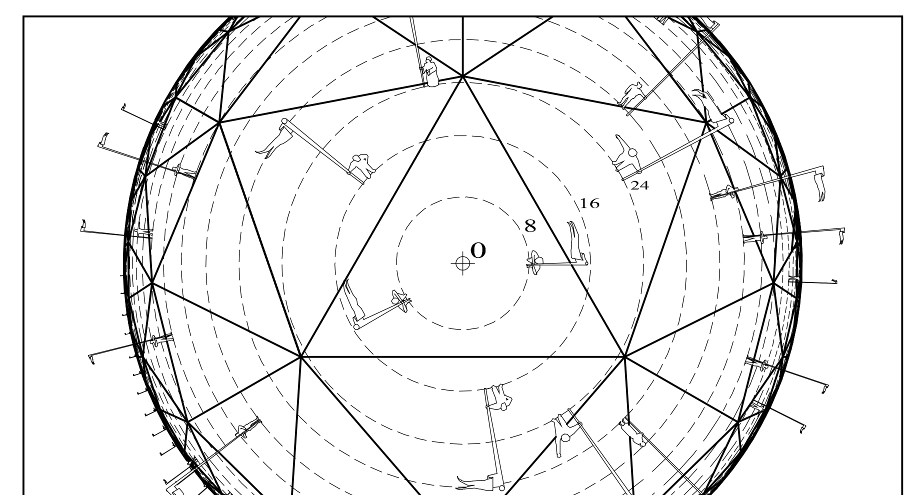
Type of Geometry | Constant factor | Distance to the Plane

Elliptic | k= 39.74173 | z = 80



Type of Geometry | Distance to the Plane

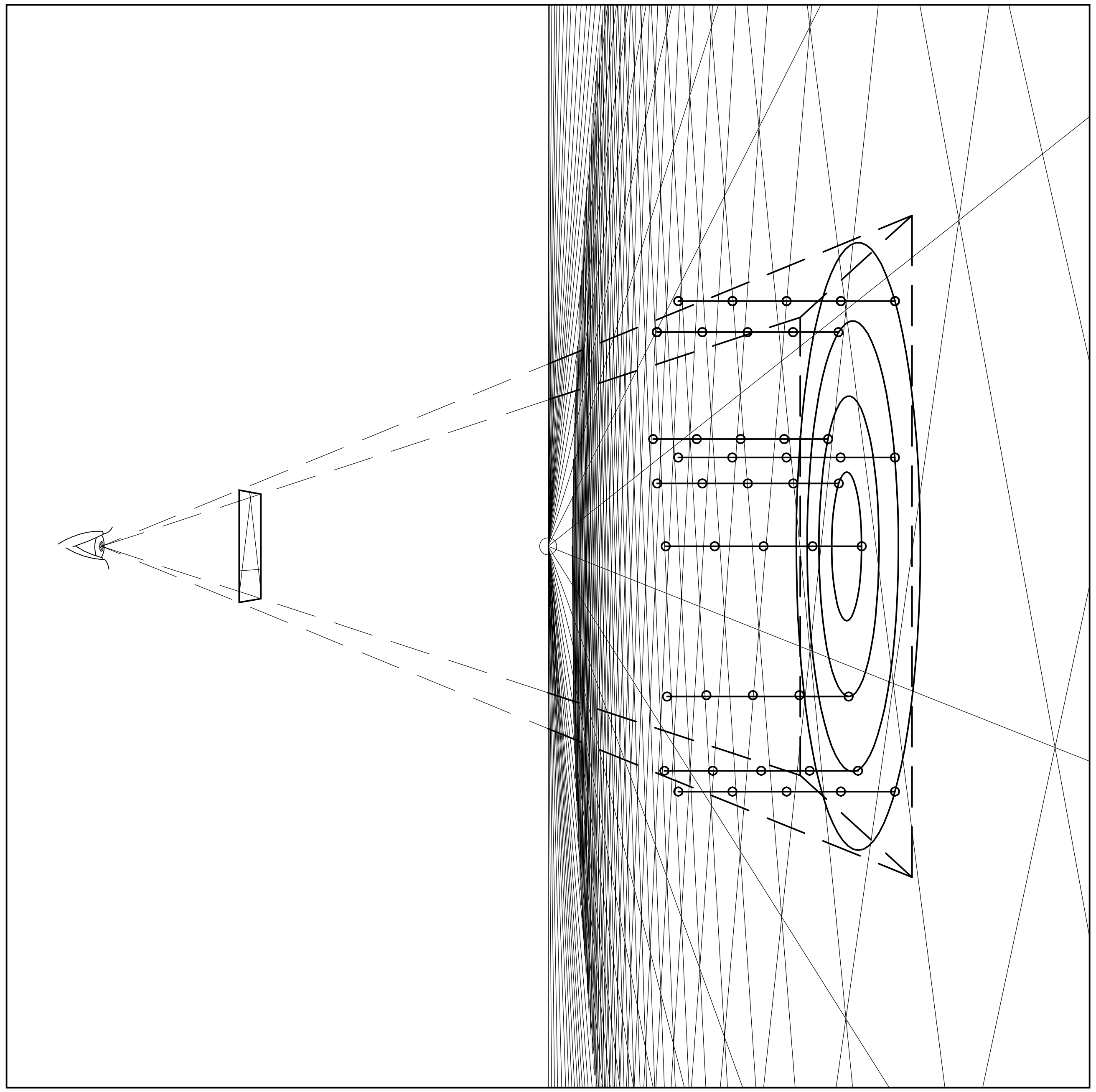
Euclidean | z = 80



Type of Geometry | Constant factor | Distance to the Plane

Hyperbolic | k= 40.34663 | z = 80

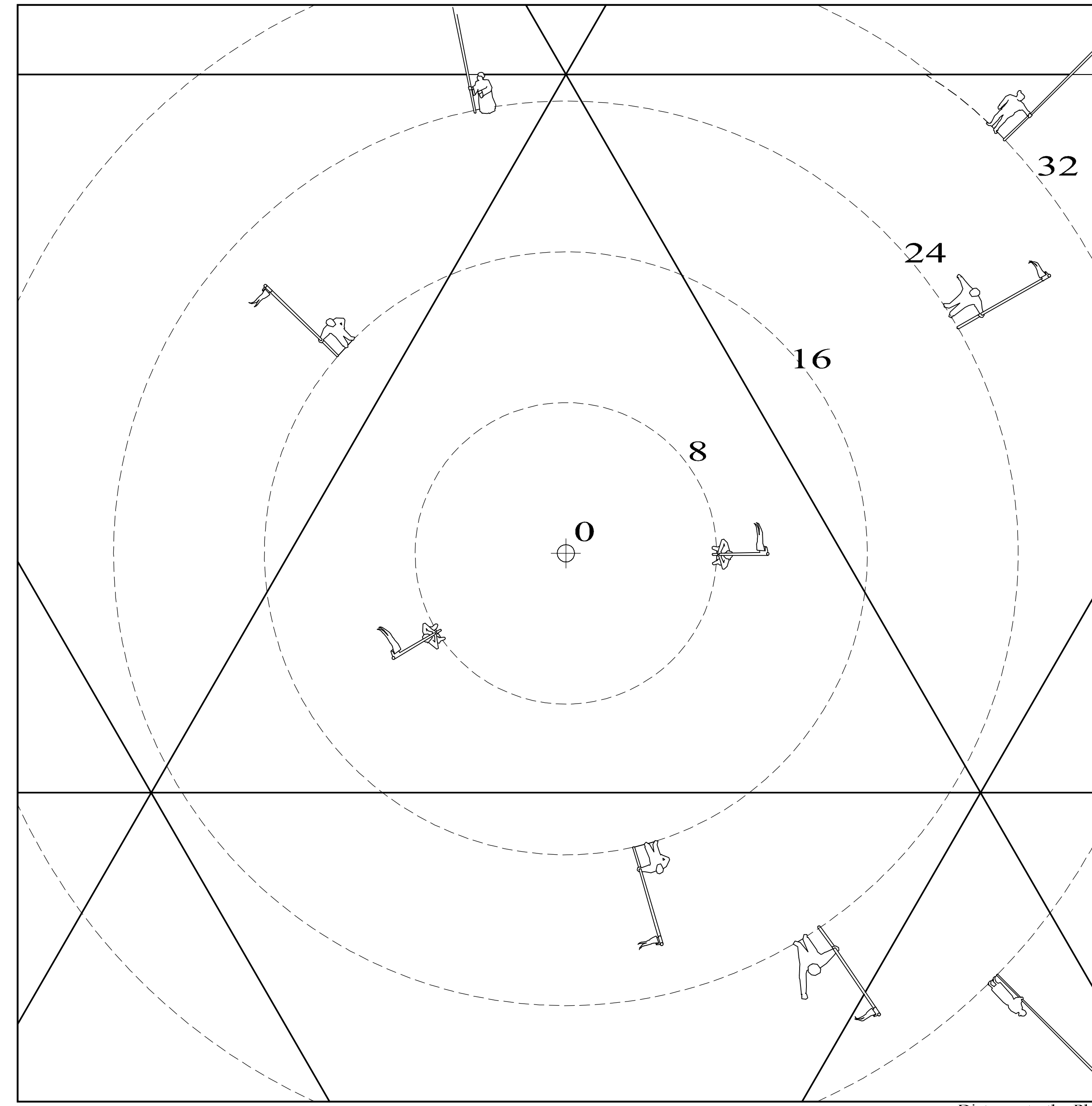
DRAWING 67: **Distance Diagram (graphed in a Euclidean space) showing the Eye, Perspective picture-plane, and a Euclidean plane with a tiling of equilateral triangles, a few of the concentric circles, and a few perpendiculars (with bubbles at five units apart).**



Euclidean "DistanceDiagram" of a Euclidean space.

Distance
to Plane: 80

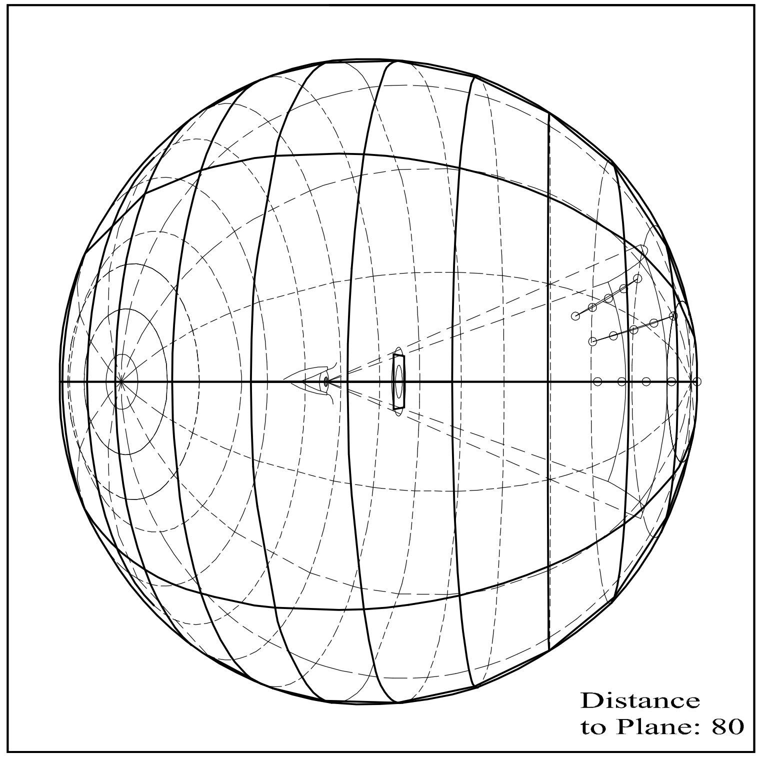
DRAWING 68: **The Perspective image of the Euclidean plane viewed in a perpendicular manner, showing a tiling of equilateral triangles, concentric circles, and various figures holding perpendicular flags 20 units high.**



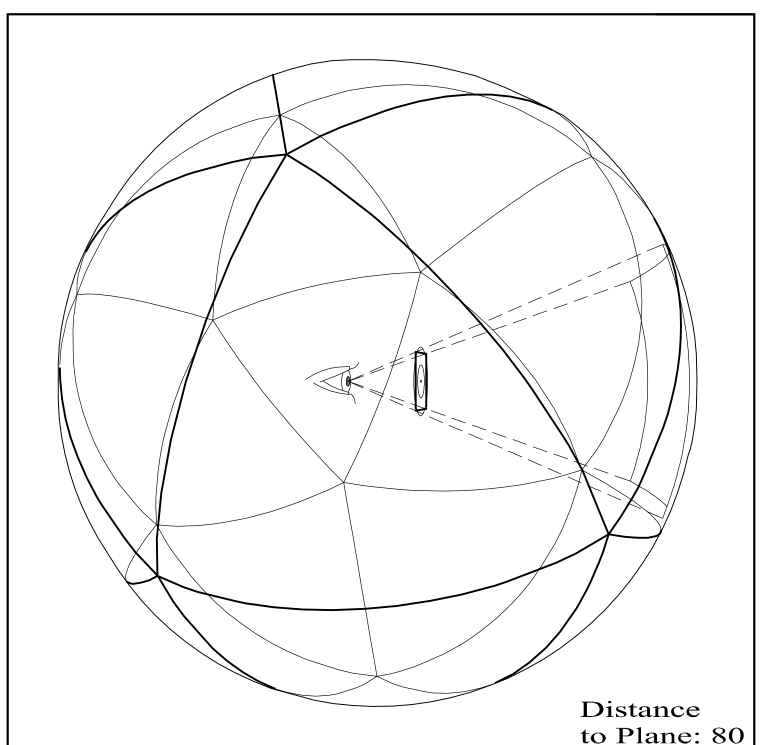
Type of Geometry
Euclidean

Distance to the Plane
z = 80

DRAWING 69: **"Distance diagram" (graphed in Euclidean space) showing two aspects of the Elliptic Plane of *Drawing 70*.**

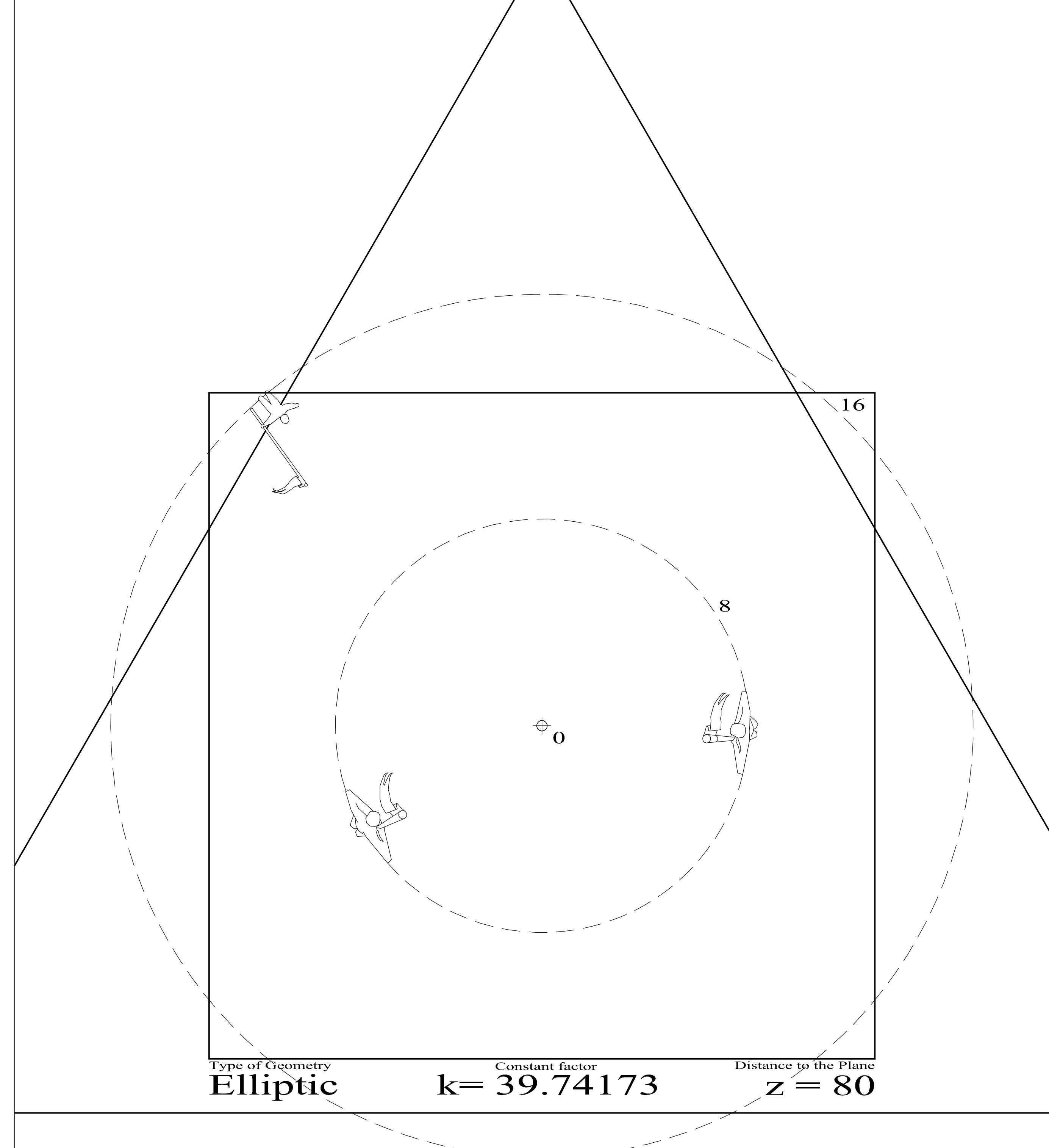


Euclidean "Distance-at-Angle Diagram":
Elliptic Plane with Concenetric Circles

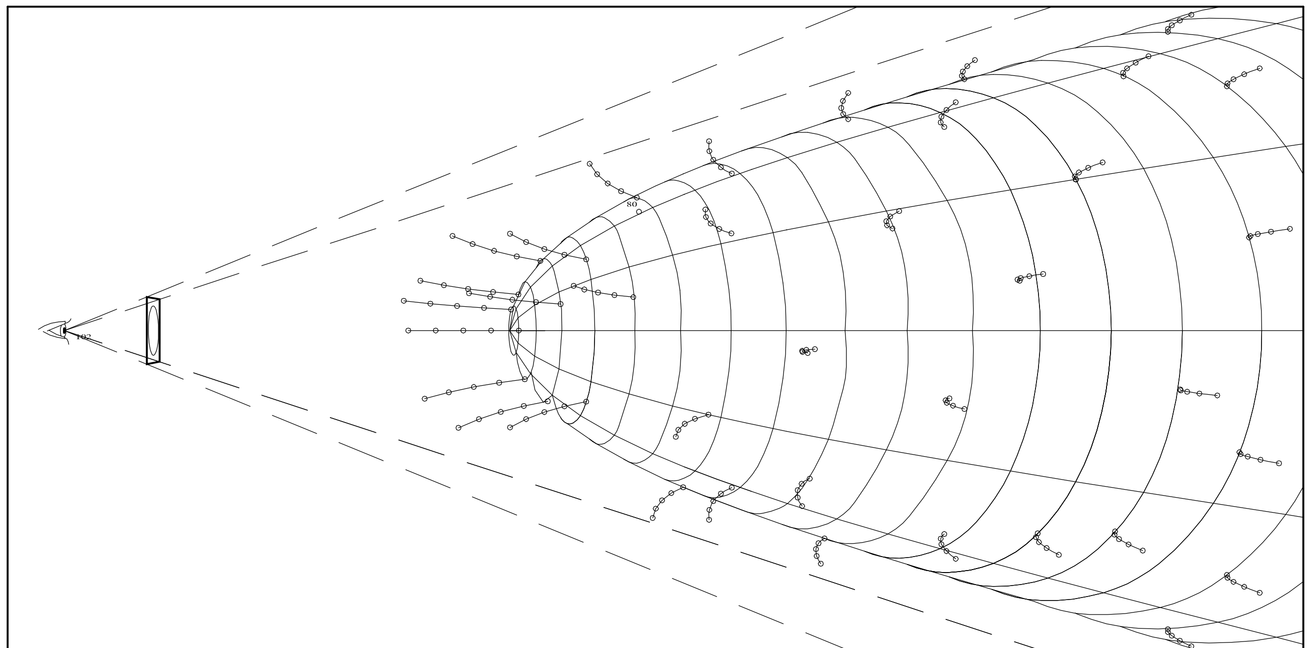


Euclidean "Distance-at-Angle Diagram":
Elliptic Plane with Tile of Equilateral Triangles

DRAWING 70: **Perspective image of an Elliptic plane viewed in a perpendicular manner, showing a tiling of equilateral triangles, concentric circles, and perpendicular figures.**

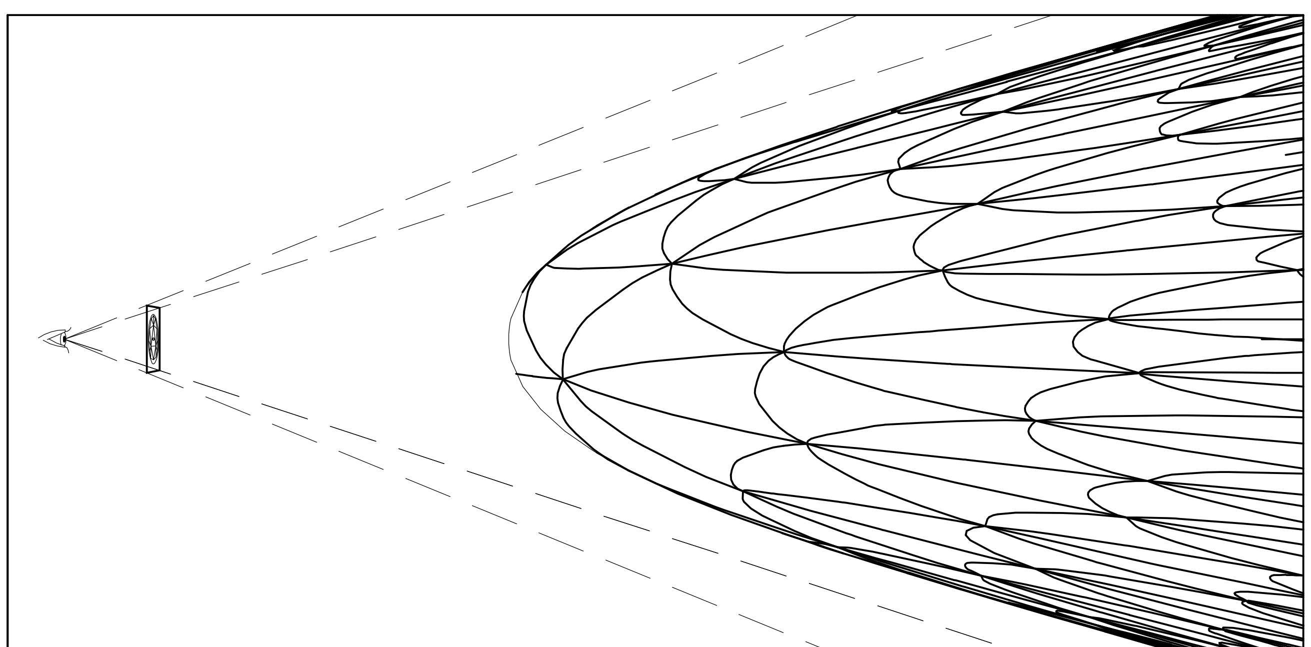


DRAWING 71: **"Distance Diagram" (graphed in Euclidean space) showing two aspects of the Perspective setup viewing a Hyperbolic Plane (the view of *Drawing 72)*.**



Euclidean "Distance Diagram" of the Hyerpbolic plane with Concenetric Circles and Perpendiculars

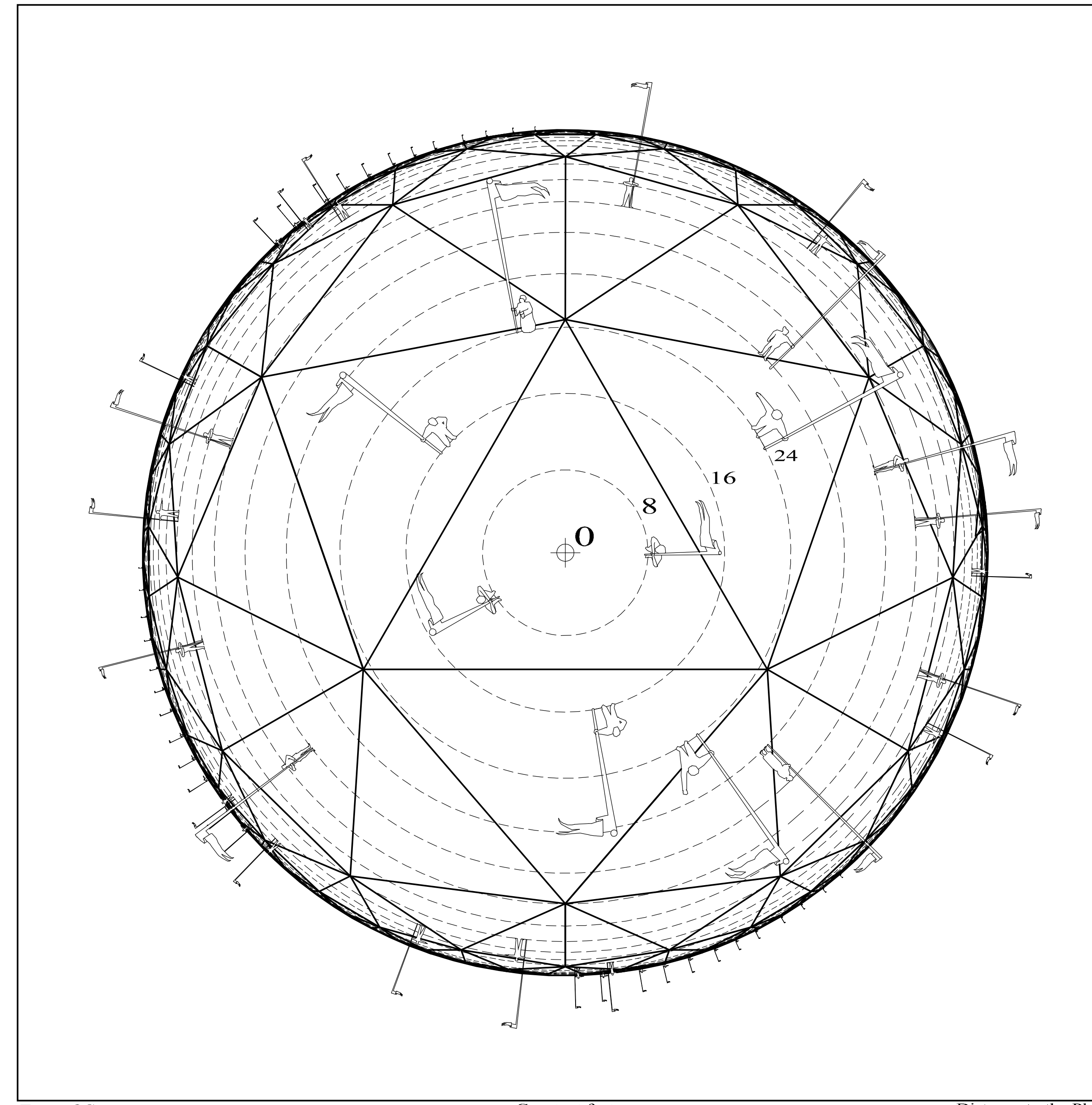
Distance to Plane: 80



Euclidean "Distance Diagram" of the Hyperbolic plane with tiling of equilateral triangles

Distance to Plane: 80

DRAWING 72: **Perspective image of a Hyperbolic plane viewed in a perpendicular manner, showing a tiling of equilateral triangles, concentric circles, and perpendicular figures (holding flags 20 units in height).**



Type of Geometry	Constant factor	Distance to the Plane
Hyperbolic	k= 40.34663	z = 80