Non-Euclidean Perspective

Part 1

An Introduction
to the
Perspective Illustration
of
Non-Euclidean Geometry

by

Jim Barnes, architect

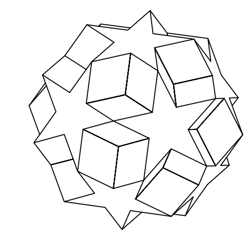

Table of Contents

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Chapter 1:
Getting Ready

2nd June 2013

To: Christopher Grubbs
Member of the American Society of Architectural Illustrators
San Francisco, California; USA

Hi Chris. At last, I'm sending these first three chapters. I hope to carry you safely across the borders of conventional logic, and show you the basic optical sights in the strange foreign spaces of Non-Euclidean Geometry. Please tell me which aspects of this tour you like, and which I might improve for future tourists.

You are going to have to trust my text with respect to mathematical method and measurement. Equations and formal proofs are omitted. My Perspective images are computer-generated, using exact data. This text also includes a few "diagrams", which are not Perspective constructions.

This text uses "*Non-Euclidean*" to refer to geometries which alter the "Fifth Postulate" of Euclid's ***ELEMENTS***. Due to the 5th Postulate we are accustomed to believing that when two perpendiculars are drawn from a base line (on a single plane), those two perpendicular lines will proceed outward at equal distances from each other, even if extended indefinitely. Non-Euclidean logic changes the Postulate and proposes that:

- the two perpendiculars converge (and eventually intersect); or,
- they diverge; or,
- their distance varies from place to place, or from case to case.

This tour is merely introductory, so we shall examine only the first two alternatives.

When the perpendicular lines converge, the new spatial logic is called ***Elliptic Geometry.*** When they diverge, we call it ***Hyperbolic Geometry***.

Incidentally: In all three geometries we continue to use Euclid's definitions: When a straight line makes adjacent angles equal to one another, each of those equal angles is called a *right angle (*measured as "*90 degrees*" or " *Pi /2 radians*"), and the straight line standing on the other is called a *perpendicular* to that on which it stands. Euclid's 4th postulate assumes: "*that all right angles will be equal to one another*", an assumption which seems to establish consistent values for angles. This 4th postulate, and our familiar angular meanings, is maintained in all the geometries of this text. In my illustrations, a *perpendicular* angle is marked by a little square (□).

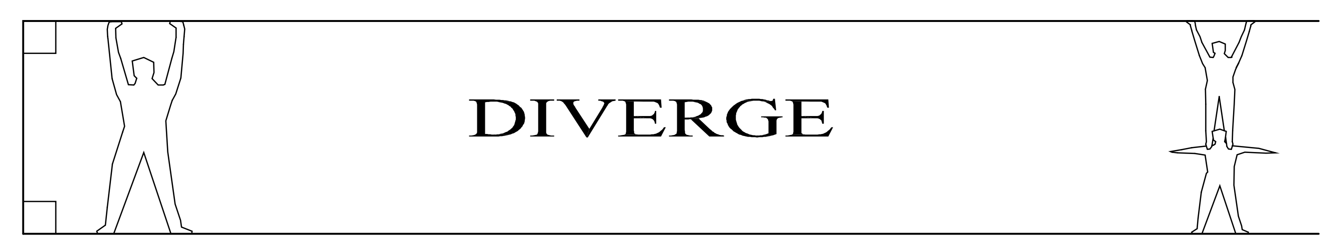
Let's begin.

DRAWING 1: **Diagram of our 3 alternate postulates**

When two straight lines are drawn mutually perpendicular to a third, as they proceed outward their distant from each other shall:

When this assumption is made, it is called **ELLIPTIC GEOMETRY.**

This is the traditional postulate of **EUCLIDEAN GEOMETRY**.



Making this assumption is called **HYPERBOLIC GEOMETRY**.

Principle #1: Within relatively small regions of space, Non-Euclidean Geometries are approximately Euclidean. At an infinitesimal dimension (at a *point*) Non-Euclidean rules devolve to Euclidean character.

I think it's helpful to distinguish between "*Approximately Euclidean*" and "*Precisely Euclidean*". In approximately Euclidean space the accuracy of measurements is limited, but they conform to Euclidean principles. If we improved the accuracy of our measurements by extending outward, to survey longer distances or by acquiring surveying instruments of finer calibration, it is possible that we would discover that our Approximately Euclidean region was actually only a small part of either an Elliptic or Hyperbolic space. Precisely Euclidean, on the other hand, presumes that it will always be strictly Euclidean, even under scrutiny of infinite accuracy. But measurement is inherently limited, so to prove that a space is precisely Euclidean is impossible. For a space to be considered precisely Euclidean requires a logical assumption, the 5th Postulate.

By this line of reasoning, you may assume that you already live in a Non-Euclidean universe and that you've been looking at Non-Euclidean Perspectives all your life. Planet Earth is too small for us to measure whether the larger cosmos is actually Elliptic or Hyperbolic. All we know is that our measurements seem Approximately Euclidean. In Approximately Euclidean regions perspective images are, for all practical purposes, identical and interchangeable.

If such subtle word distinctions seem like nothing more than silly semantics, fear not, because we are next going to use this principle as the passport for our imaginary journey -- it will provide the logic of a realistic Perspective "picture plane".

From here forward in this text, for the sake of brevity, I will use the word *Euclidean* to mean *Precisely Euclidean*, and *Approximately Euclidean* will keep its longer name.

Next, I will try to define for you our Perspective apparatus.

DRAWING 2: **"Approximately Euclidean" space --**

Three perspective views, where the size of the region of space viewed may be considered as relatively infinitesimal.

Type of Geometry
Elliptic

Constant factor
k = very large compared to the size of the region

Type of Geometry
Euclidean

Type of Geometry
Hyperbolic

Constant factor
k = very large compared to the size of the region

Our Perspective apparatus:

Several methods could provide Perspective views of Non-Euclidean Geometries on approximately Euclidean surfaces. The simplest imagines a vast space, far away, in which we build astronomically large geometric figures. We adopt the logical assumption that light travels in straight lines. We position our Eye directly in front of a window. Our Perspective drawing is the collection of points where light rays intersect that transparent plane. Our picture plane will not deform (to any perceptible manner) when we switch postulates from precisely Euclidean to Elliptic, or to Hyperbolic. Of course, switching the geometry of the universe is purely hypothetical.

The adjacent diagram shows our Perspective apparatus. A central line of sight is set perpendicular to a flat picture-plane. This as the basic geometrical setup of traditional Italian Renaissance Perspective art. It's shortest formal name is: ***gnomonic*** projection. Let me jump ahead and reveal a significant conclusion:

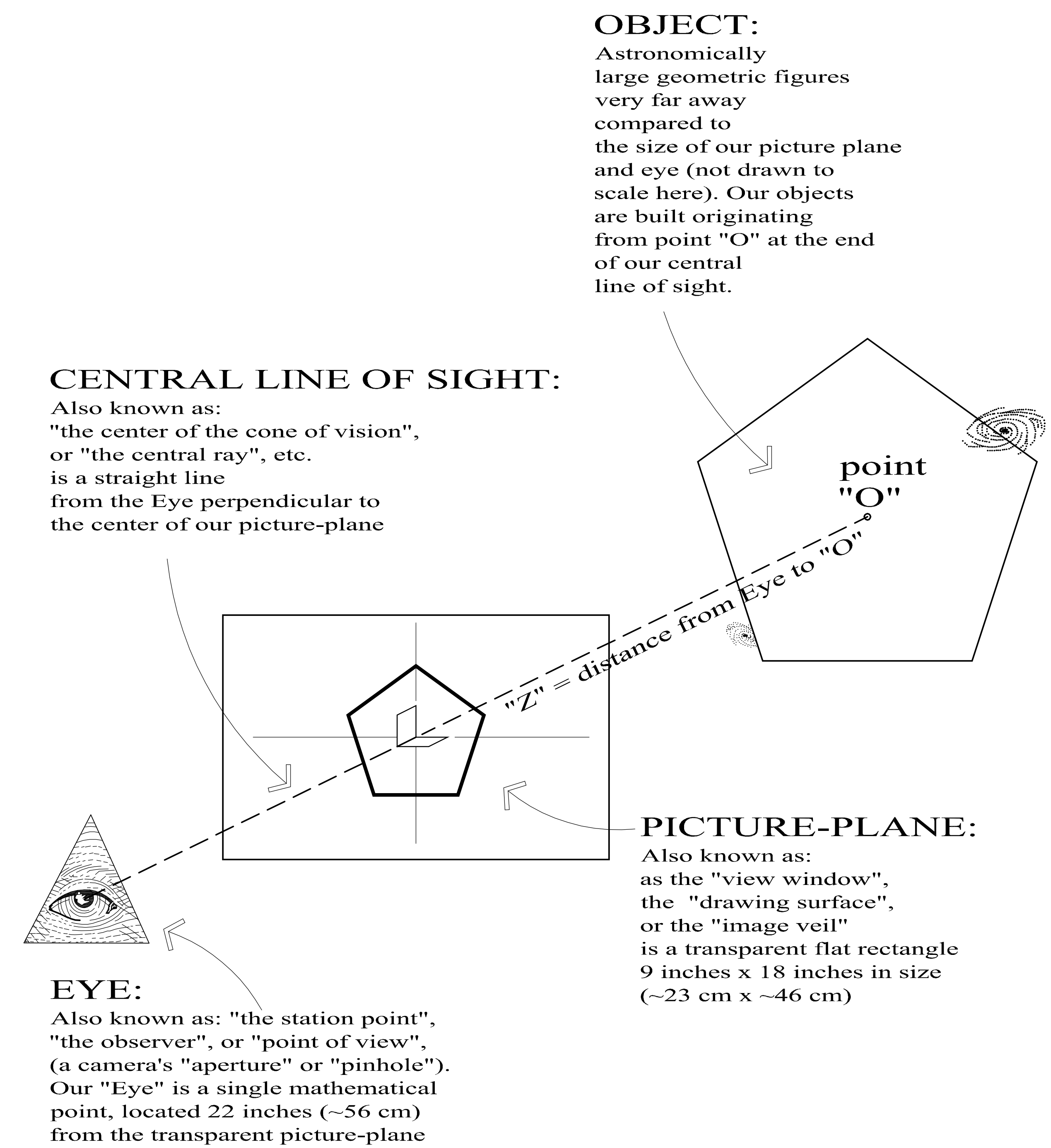
Principle #2: **Straight lines always appear straight in *gnomonic* Perspective.**

This remains true whether our geometry is Elliptic, Euclidean, or Hyperbolic. Other perspective formats exist, though seen less often. None renders all straight lines as straight. Surprisingly, whether the human eye sees straight lines as straight is a subject of academic discussion, a minor but ancient controversy. Let us proceed by means of the traditional gnomonic Perspective method, the universal picture-making format now employed worldwide.

So, we now adopt the gnomonic Perspective format, as diagrammed here. We will call simply "Perspective". Straight lines will appear straight. Until our Perspective apparatus is reconsidered, my language may become somewhat loose as I permit myself to speak as if the images of Perspective drawing, human vision, pinhole photography, and the camera-obscura were all identical, equivalent, and interchangeable.

Hereafter, in this text, the word ***line*** will be used to mean *straight line*, and ***curve*** denotes a non-straight line.

DRAWING 3: **Diagram showing our Perspective-making format**



Chapter 2:

The Saccheri Quadrilatereal and Image Size

The oldest figure in Non-Euclidean Geometry is called **the *Saccheri Quadrilateral***. We build ours thus: far beyond our Perspective picture-plane, we establish a new figure plane perpendicular to our central line of sight. The point where the central line of sight meets this plane we call "O" ("origin"). From "O" we construct on the figure-plane a base line AB. From that base line we raise two equal perpendiculars , AC and BD. Their ends are then connected with a top line, CD. We ask: for each of our three alternate versions of the 5th Postulate, what happens to the length of top line, CD, and to the new interior angles at C and D?

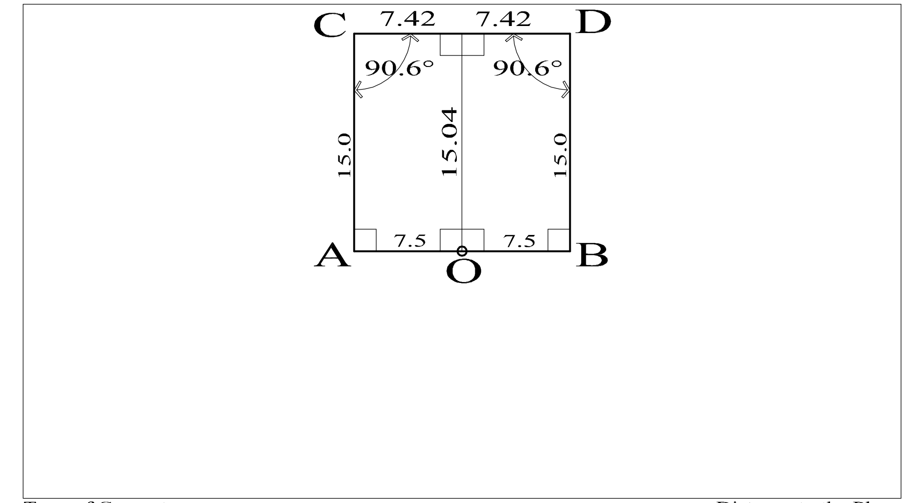
Please notice that in Elliptic space the length of the top line, CD, is measured as being shorter than base line, AB, below. This conforms to our postulate for Elliptic Geometry, that two such perpendiculars will *converge*. Likewise Euclidean space shows the length of line CD equal to AB, and the Hyperbolic Perspective notes that top line CD is longer than base line AB. Our measurements match our assumptions. But in all three Perspective images, the length of the top line remains visually equal to the length of the base below. The explanation I use is: ***in Elliptic Perspective, as we build outward, space appears to stretch.*** As space becomes less dense, distance seems to inflate and measuring units appear longer. Since the top line, CD, is built farther away from origin "O" than the base line AB, its measurement appears elongated -- the space appears stretched.

For the opposite axiomatic assumption, my explanation then reverses: ***In Hyperbolic Perspective, as we build outward, space appears to grow ever denser.*** The paradox is that while Elliptic space appears to be stretching, and Hyperbolic space appears to be compressing, our measuring rulers are unaltered. The difficult concept to explain is that the space is simply decreasing, or increasing, in quantity.

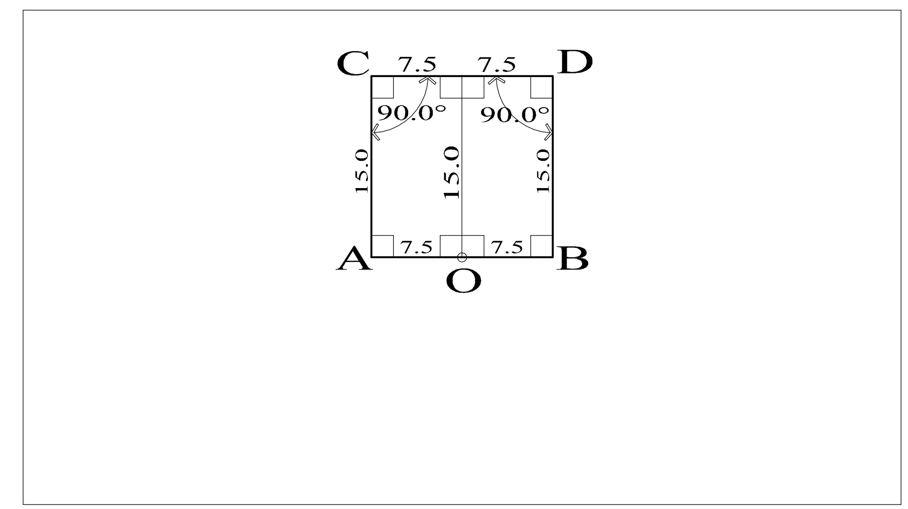
This probably seems strange and confusing. You need to see more images. Let us push onward. You may then rethink this explanation.

The obvious difference between these three images is their size. Let us next focus on that.

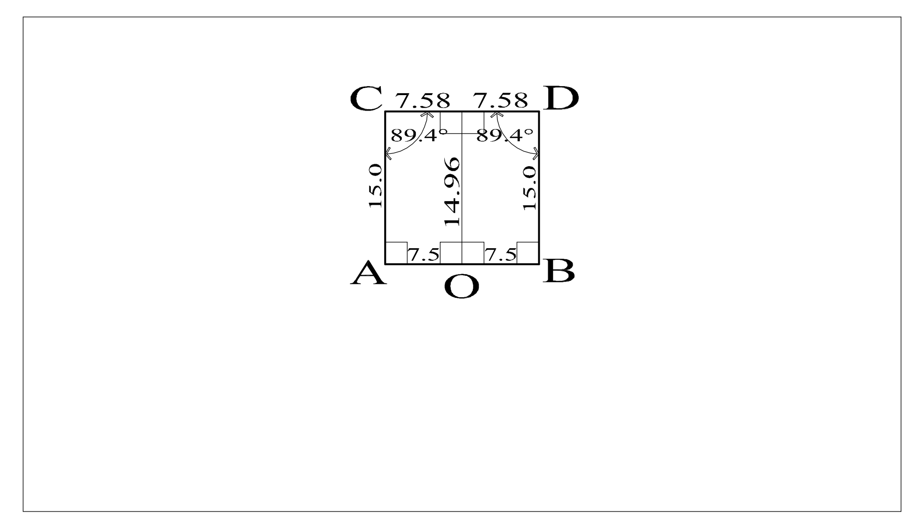
DRAWING 4: **Perspective views of a Saccheri Quadrilateral**



Type of Geometry Elliptic | Constant factor k = 100 | Distance to the Plane z =100



Type of Geometry Euclidean | Distance to the Plane z =100



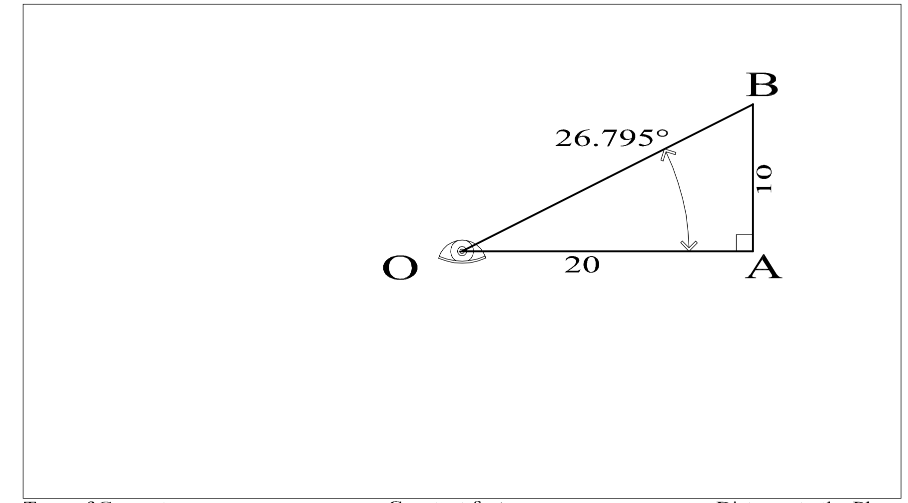
Type of Geometry Hyperbolic | Constant factor k = 100 | Distance to the Plane z =100

Principle #3: When viewed at equal distances, any given length will appear larger in Elliptic, and smaller in Hyperbolic, than in Euclidean space.

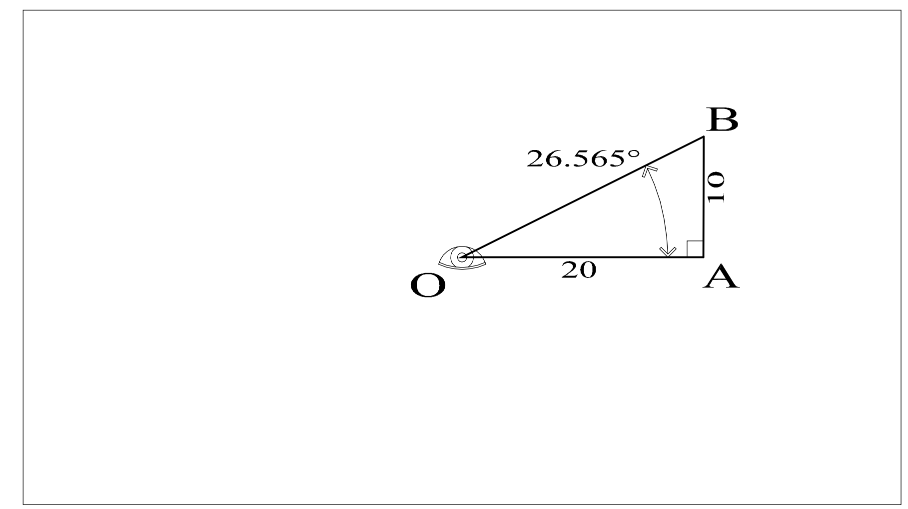
Since our hypothetical universe is consistently either Elliptic, Euclidean, or Hyperbolic, we can use the accompanying three Perspective images as scaled "section views" of our Perspective apparatus. At equivalent conditions Perspective images in Elliptic space will always appear smaller than in Euclidean space, and Perspective images in Hyperbolic Geometry will always fill smaller angles of view than in Euclidean space.

You may perhaps recall a Euclidean theorem that says the sum of the angles of any triangle equals two right triangles (180°). That rule is valid only in Euclidean Geometry. In Elliptic Geometry, the sum of the interior angles of a triangle will always be *more* than 180° and in Hyperbolic space a triangle's interior angles always have a sum *less* than 180°. Our Perspectives in *Drawing 5* are consistent with these rules .

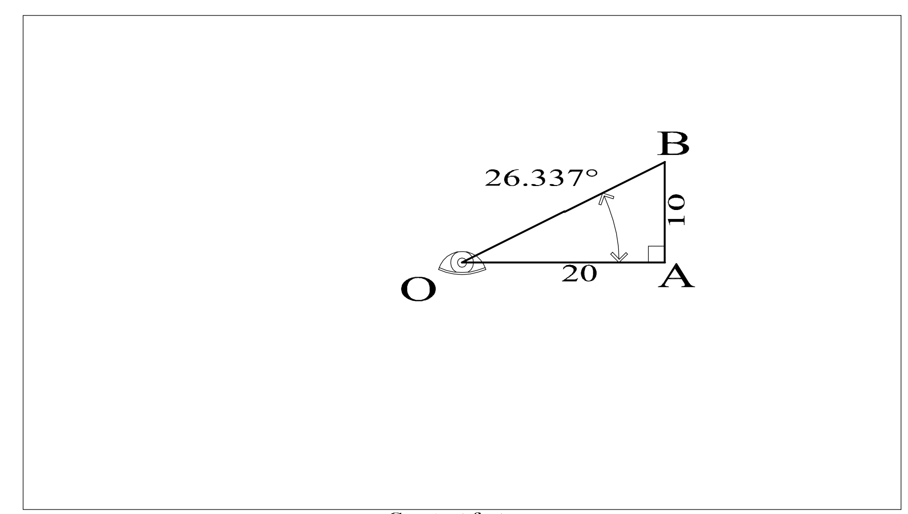
DRAWING 5: **Perspective views of a triangle.**
(The Eye in these illustrations is merely diagrammatical.)



Type of Geometry: Elliptic — Constant factor: k = 100 — Distance to the Plane: z =100



Type of Geometry: Euclidean — Distance to the Plane: z =100



Type of Geometry: Hyperbolic — Constant factor: k = 100 — Distance to the Plane: z =100

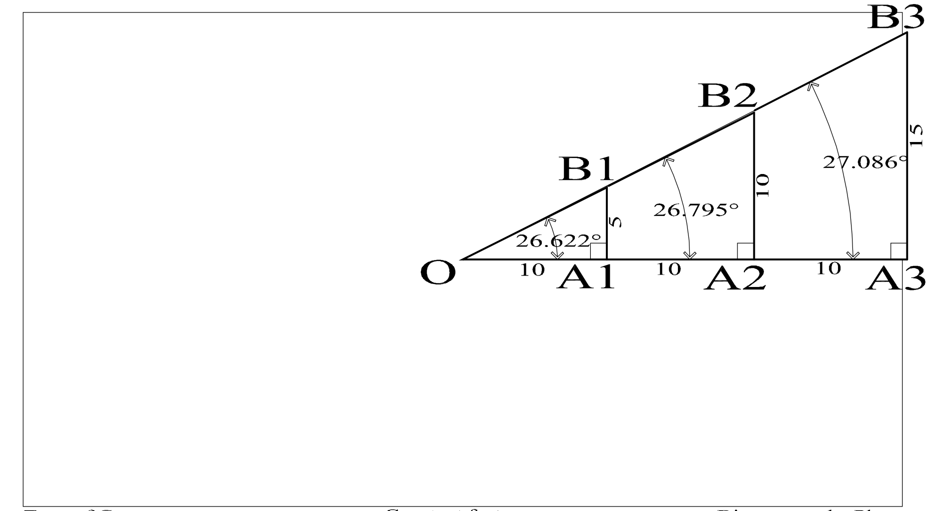
Please permit me to extend the triangles.
When we double the lengths of both legs adjacent to the right angle, then:
-- in Elliptic geometry, the angles at the remaining two corners become larger and larger;
-- in Euclidean Geometry, the angles of the remaining two corners stay equal; while
-- in Hyperbolic Geometry, the angles become smaller and smaller.

There is a principle in Euclidean Geometry of ***similar*** triangles, that triangles with equally proportioned sides will consequently have equal angles, regardless of size. That notion of *similar* figures is invalid in Elliptic and Hyperbolic Geometries.

Drawing 6 makes a pretty good illustration of my explanation. Compared to our familiar Euclidean world, as we build outward in Elliptic Geometry, the space appears to be becoming less dense (so lengths seem to expand and their angles of view increase), while in a Hyperbolic universe, space appears to be growing ever denser (so image sizes appear compressed and the angles within which they are viewed decrease).

Let me next explain the "**k**" factor.

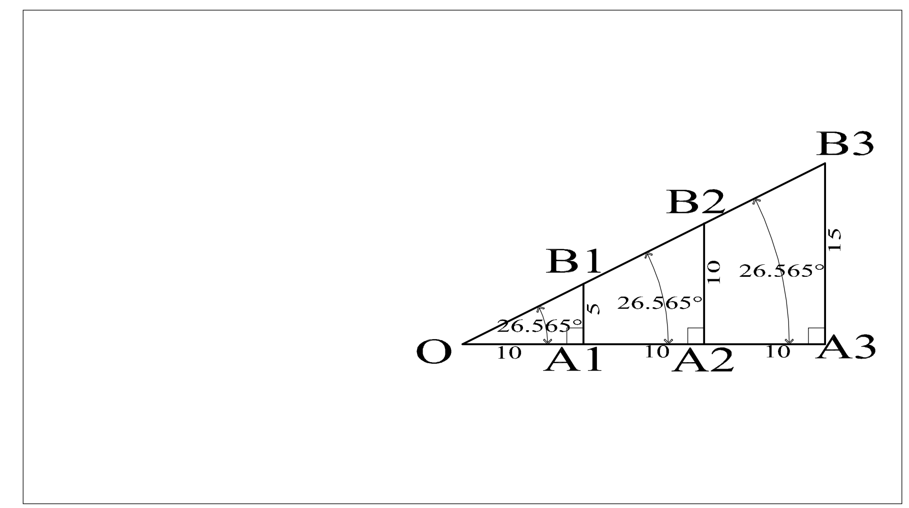
DRAWING 6: **Perspective views of extending triangles.**



Type of Geometry
Elliptic

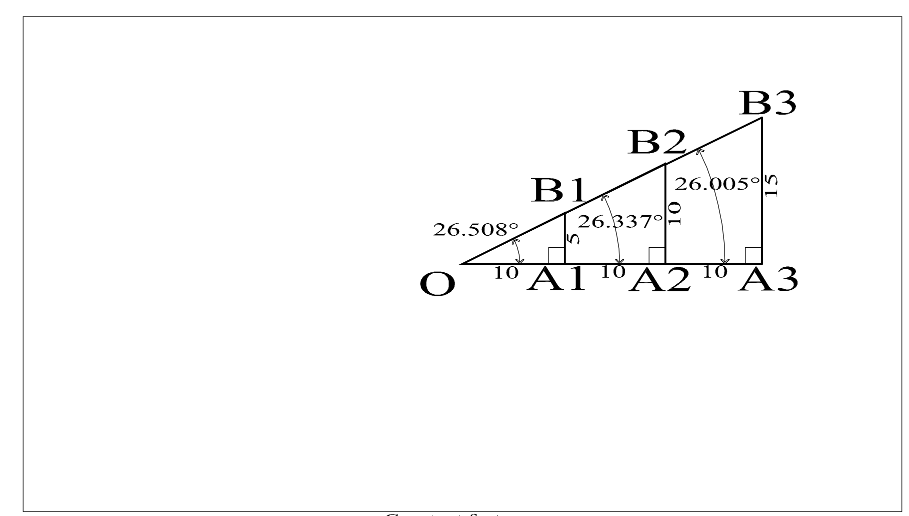
Constant factor
k = 100

Distance to the Plane
z =100



Type of Geometry
Eucldiean

Distance to the Plane
z =100



Type of Geometry
Hyperbolic

Constant factor
k = 100

Distance to the Plane
z =100

Reducing the value of the **"k" factor** by half, the previous Perspectives of *Drawing 6* now appear in *Drawing 7*.

"k" is a numerical constant regulating the rate at which the distances between two perpendiculars of our Non-Euclidean postulates will converge, or diverge. Those rates increase as the number"k" decreases.

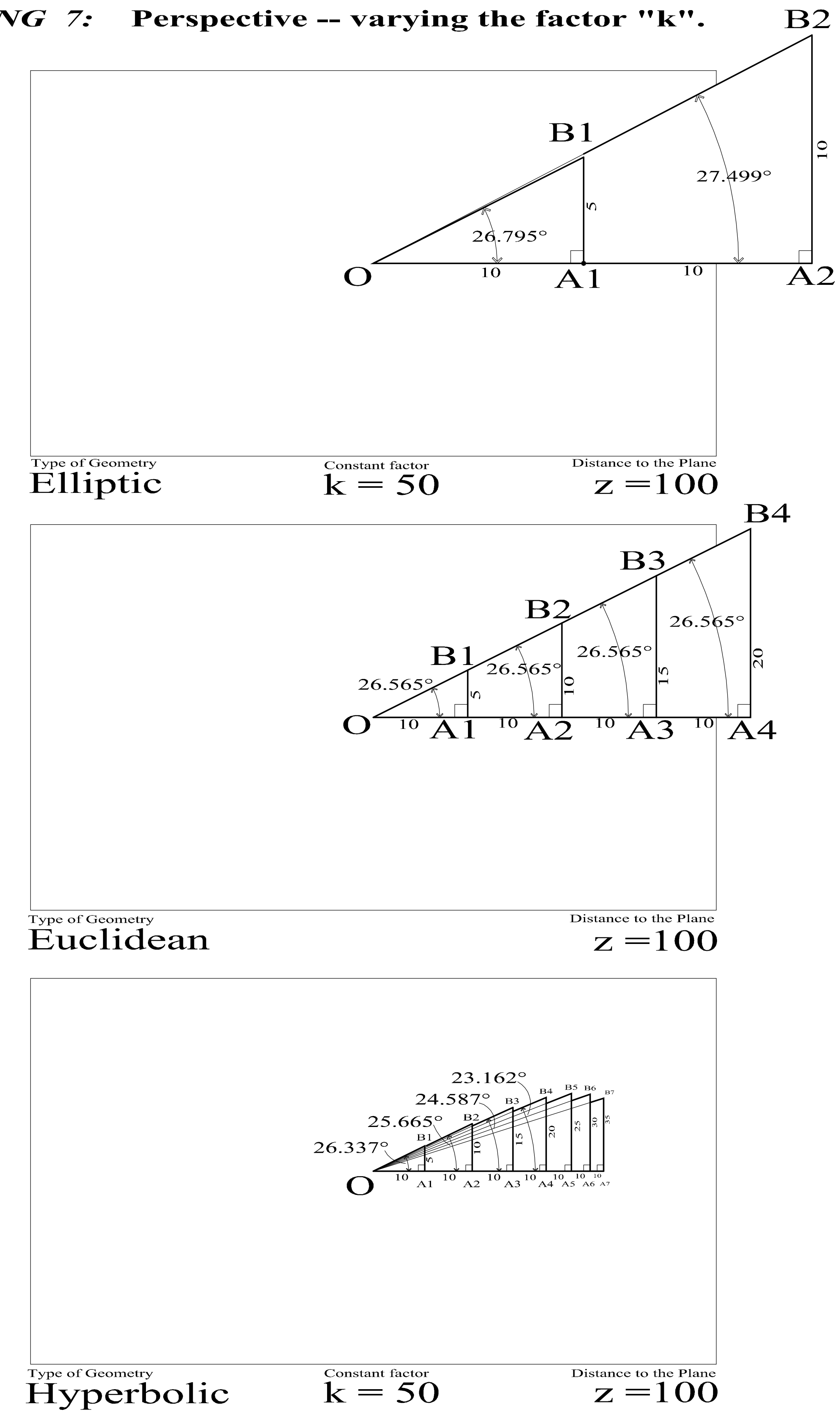
Here, in *Drawing 7* I have cut the "k" factor to half of its value for *Drawing 6*. In the resulting views, please note that the measurements of lengths and angles has been altered; and the size of the Perspective images of the Non-Euclidean figures has been changed.

There is no "k" factor in Euclidean Geometry.

The rate of ***convergence*** or ***divergence*** is described by this constant named "k". In this text we will consider conditions where there is only one "k" factor throughout any Elliptic, or a Hyperbolic, universe. But since there are infinite possible values of "k", both Elliptic and Hyperbolic Geometries are actually families having an infinite number of possible members. Each individual geometry will have its own unique "k" factor.

Beyond the scope of this text, it is here noted that it is possible for a single Elliptic or Hyperbolic space to vary the values of "k" from place to place, or case to case, without violating its version of the 5th Postulate. For the remainder of this text, however, for the sake of introductory simplicity, we will assume that our spaces are homogeneous and consistent, with the same "k" value at every point.

DRAWING 7: **Perspective -- varying the factor "k".**



Let me return now to our first Perspective, *Drawing 4*, which introduced the Saccheri Quadrilateral, and show how it appears extended outward in multiples of lengths.

Non-Euclidean quadrilaterals can have right angles at three of their interior corners, but never at all four. "Squares" and "rectangles" therefore exist only in Euclidean Geometry. Quadrilaterals with four equal sides are possible in Non-Euclidean Geometry and their corners would have four equal angles, but those equal angles could never be exact *right angles*.

It is therefore impossible to fabricate a ***Cartesian coordinate system*** in any Non-Euclidean Geometry.

Comments regarding Pages 20 and 21 (ahead):

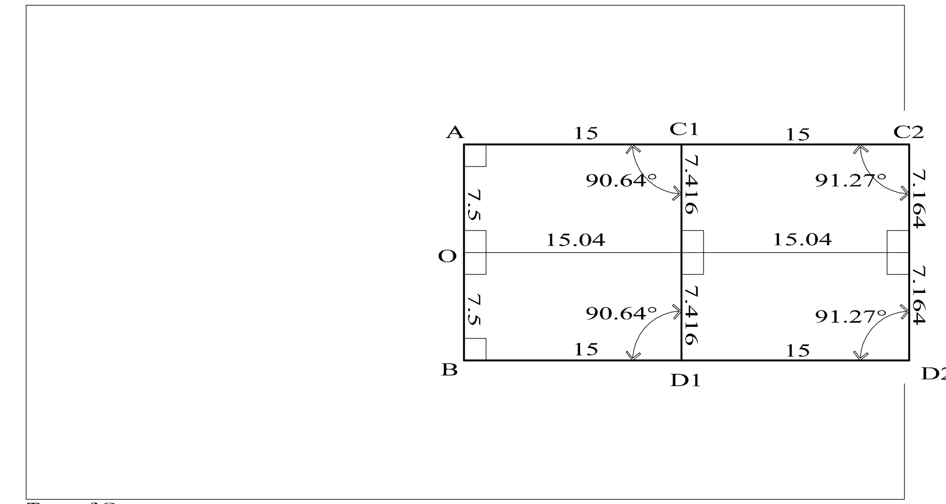
The next pair of pages show a Sacherri Quadrilateral moved around on its figure-plane. The first illustrations (*Drawing 9*) use k=100, and the second (*Drawing 10*) reduces "k" to 50.

The shape of the visual image is altered merely by the figure's position on the plane. I imagine that you may find this rather disconcerting.

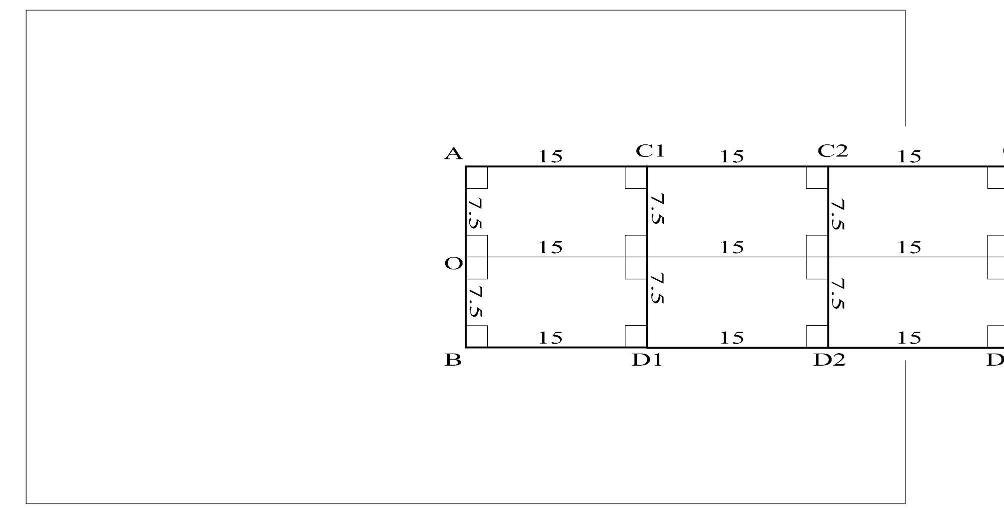
Please notice that the quantitative character of the Quadrilateral is unaltered. The figure remains rigid: neither the length of its sides, nor the measure of its angles, is changing. Only the shape of our visual Perspective changes as the figure slides around.

Chapter 3 will next examine the figure-plane as a whole, and explain these peculiar visual characteristics.

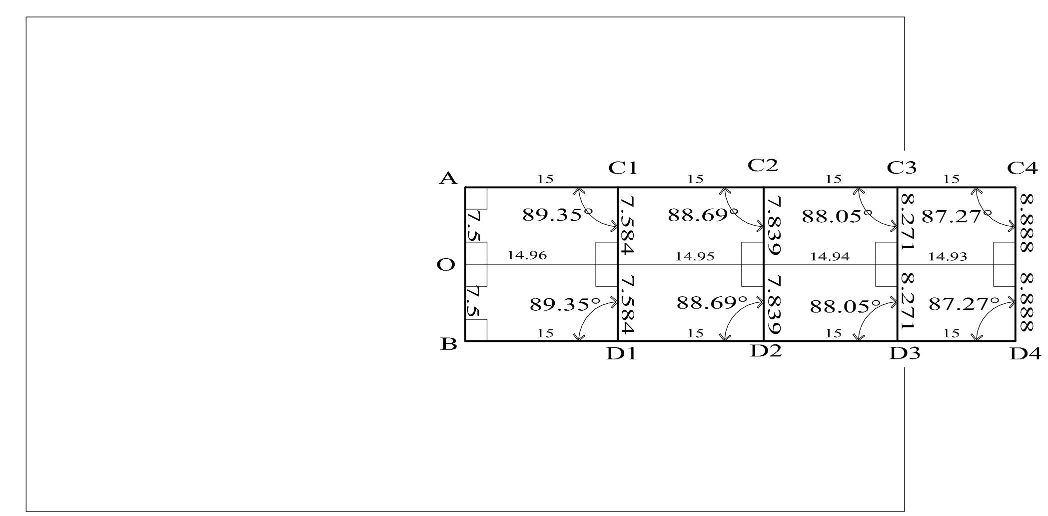
DRAWING 8: **Perspectives of extended Saccheri Quadrilaterals**



Type of Geometry Constant factor Distance to the Plane
Elliptic k = 100 z =100

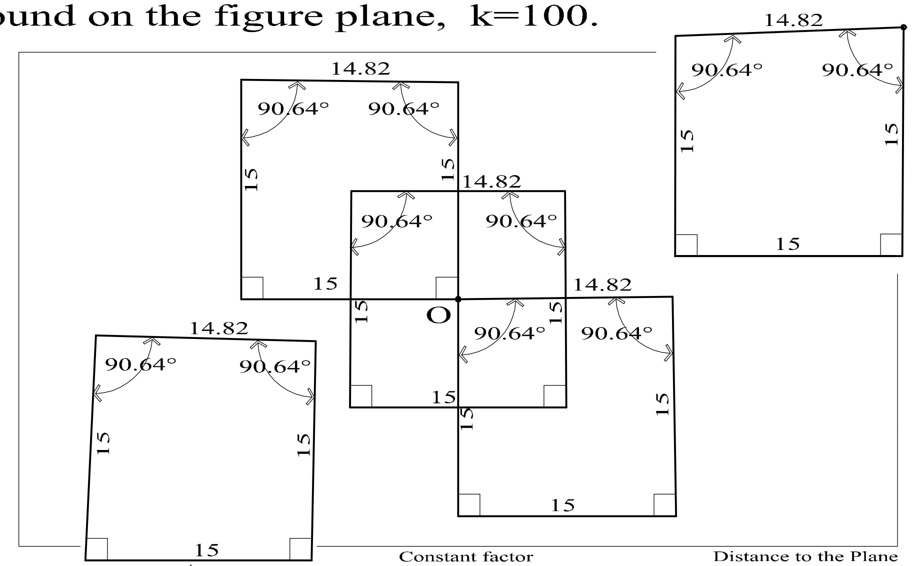


Type of Geometry Distance to the Plane
Euclidean z =100

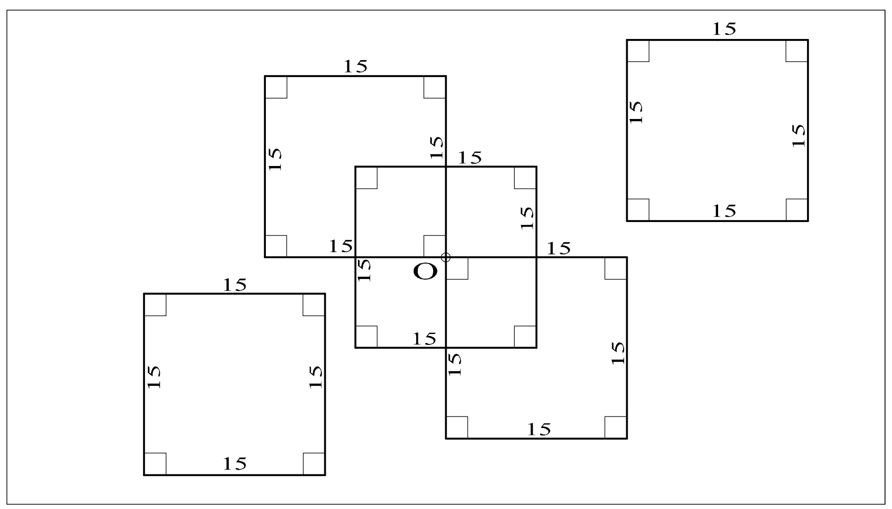


Type of Geometry Constant factor Distance to the Plane
Hyperbolic k = 100 z =100

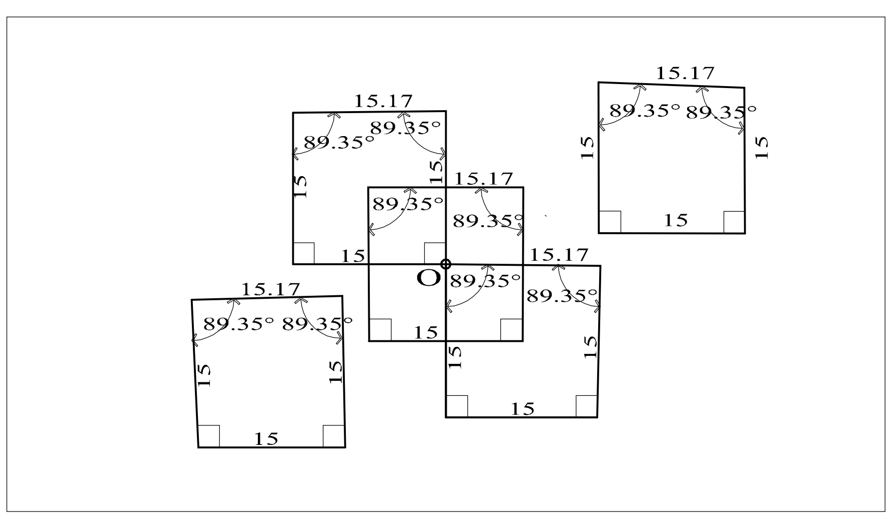
DRAWING 9: A Saccheri Quadrilateral sliding around on the figure plane, k=100.



Type of Geometry
Elliptic
Constant factor
k = 100
Distance to the Plane
z =100

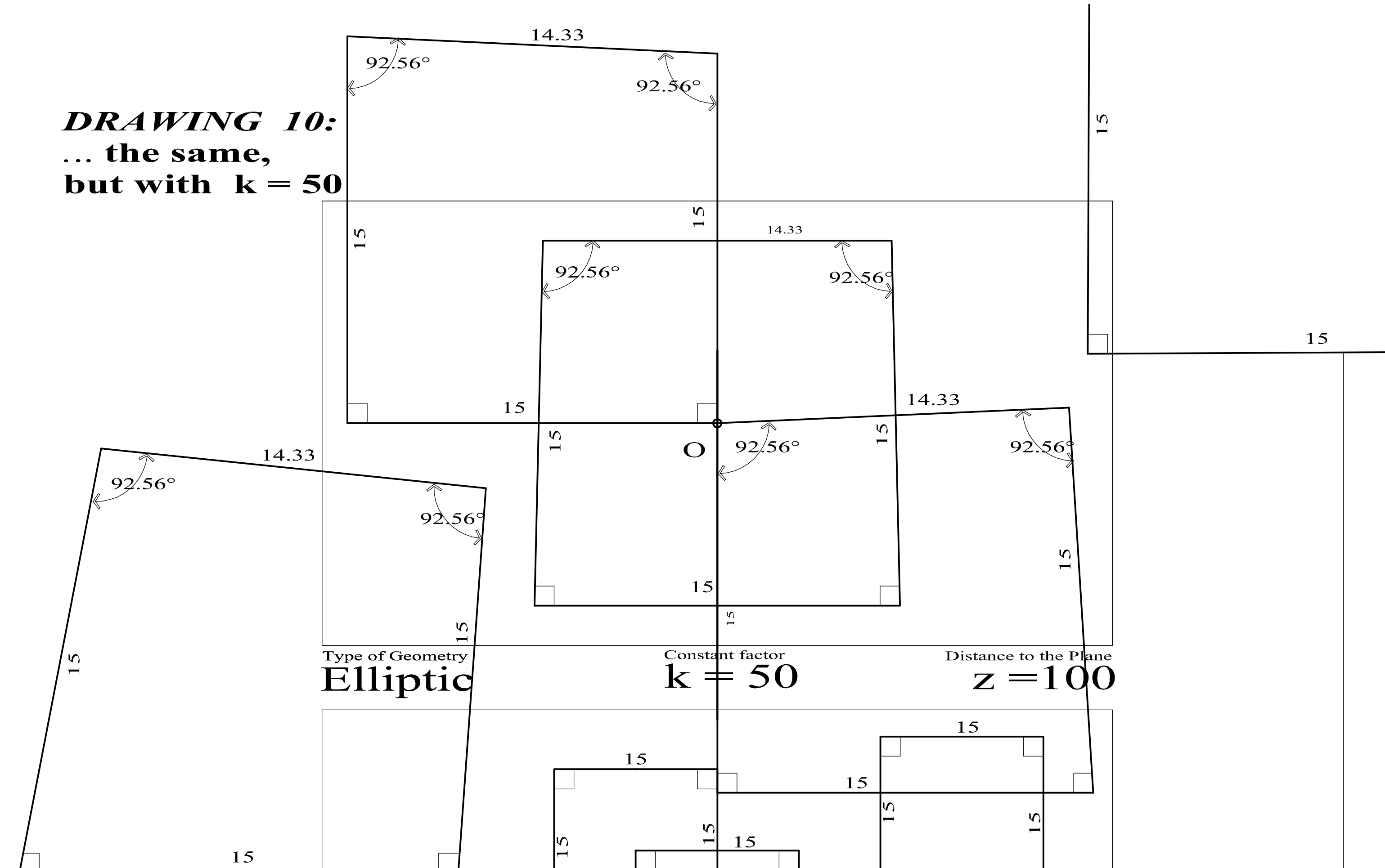


Type of Geometry
Euclidean
Distance to the Plane
z =100

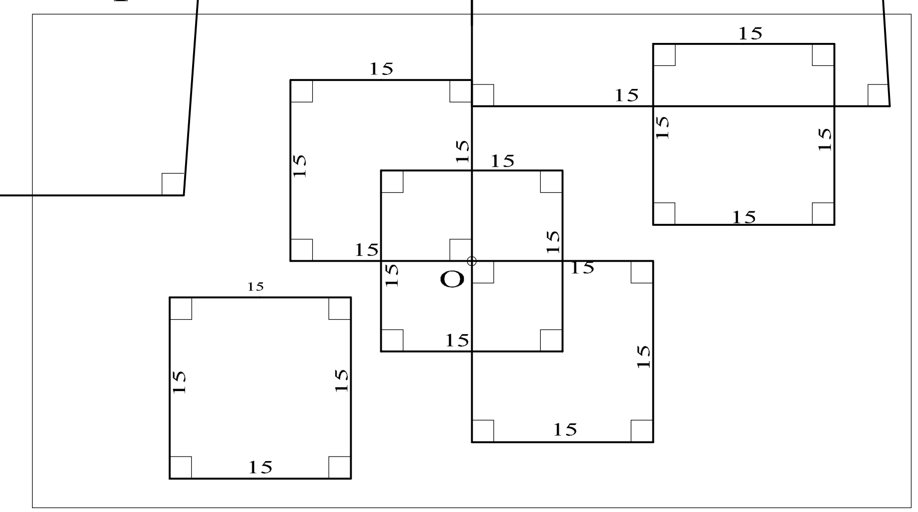


Type of Geometry
Hyperbolic
Constant factor
k = 100
Distance to the Plane
z =100

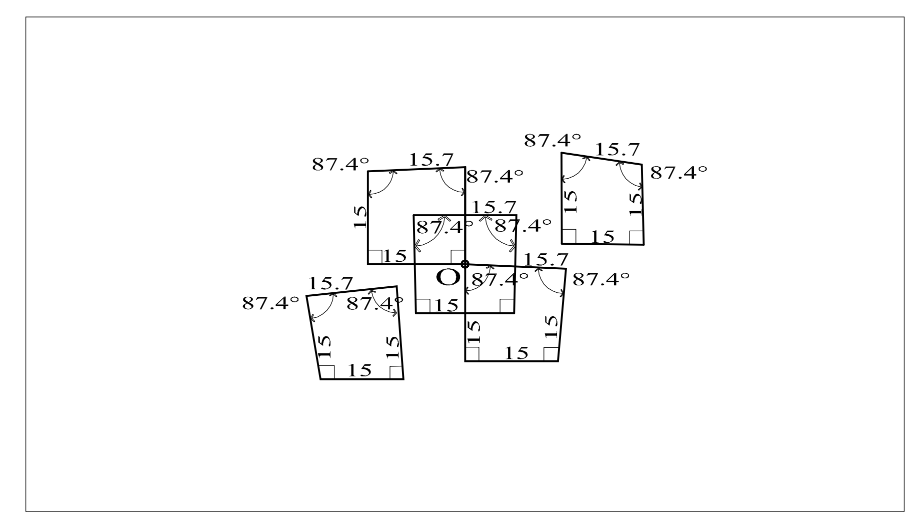
DRAWING 10:
... the same,
but with k = 50



Type of Geometry
Elliptic
Constant factor
k = 50
Distance to the Plane
z =100



Type of Geometry
Euclidean
Distance to the Plane
z =100



Type of Geometry
Hyperbolic
Constant factor
k = 50
Distance to the Plane
z =100

Chapter 3:
The Appearance of Non-Euclidean Planes

In this chapter we examine the visual character of Perspective illustrations of a Non-Euclidean *plane* (a flat *surface, which lies evenly with the straight upon itself)*. As before, we establish a plane at right angles to our central central line of sight, with the central ray of vision striking that plane at point "O" , located distance "Z" from the Eye. From point "O" this plane extends perpendicularly outward endlessly in all directions.

Since a Cartesian grid of squares is impossible to build in any Non-Euclidean space, I struck upon the idea of spinning a series of concentric circles to fill the plane. They are fast to draw - simply compute each radius and spin it full circle around point "O".

The next five pages are Perspective views of such a plane, using five values of "k". (k = 1,000, 240, 100, 50 , and 33)

Explanation of annotation numbers:

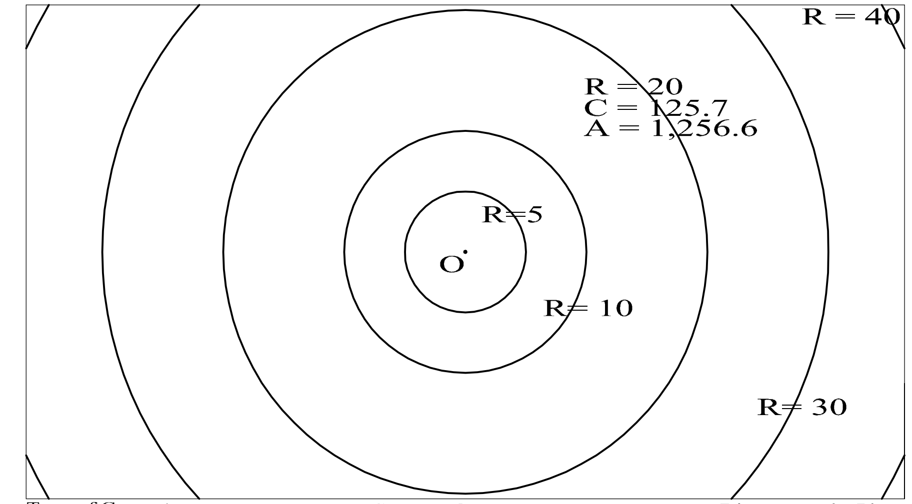
R = Radius
C = Circumference
A = Area

"R" is a percentage of "**Z**", the perpendicular distance from our Eye to "O". (For example "R=20" means that this particular circle has a radius equal to 20% of "Z"). "C" is also a length; and "A" is a square-unit value of lengths, measuring area.

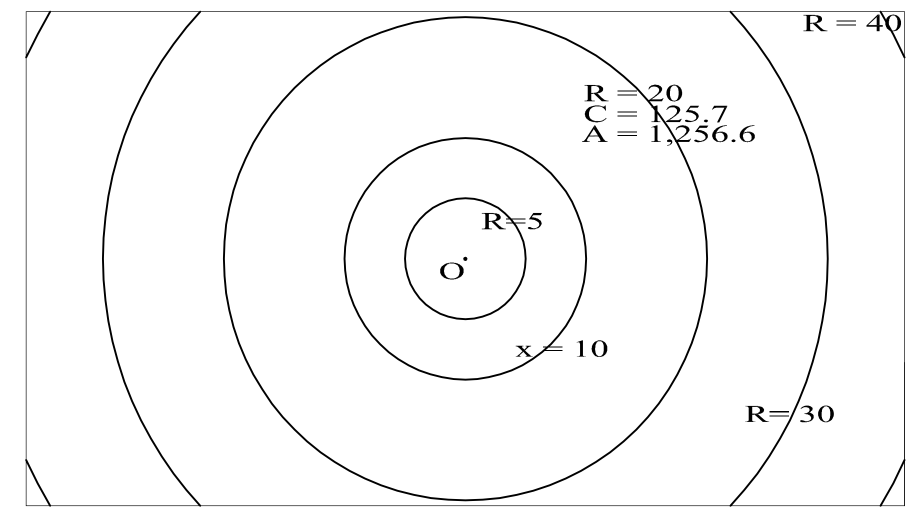
We started with an assumption that "Z" is "far away", and that these circles would "astronomically large" in size (compared to the size of our picture-plane and local viewing environment); but since we are dealing with only a single point as our Eye we may keep our geometry purely proportional in scale and need not necessarily specify any exact unit of length for our hypothetical distances.

Our discussion will continue after the next five drawings.

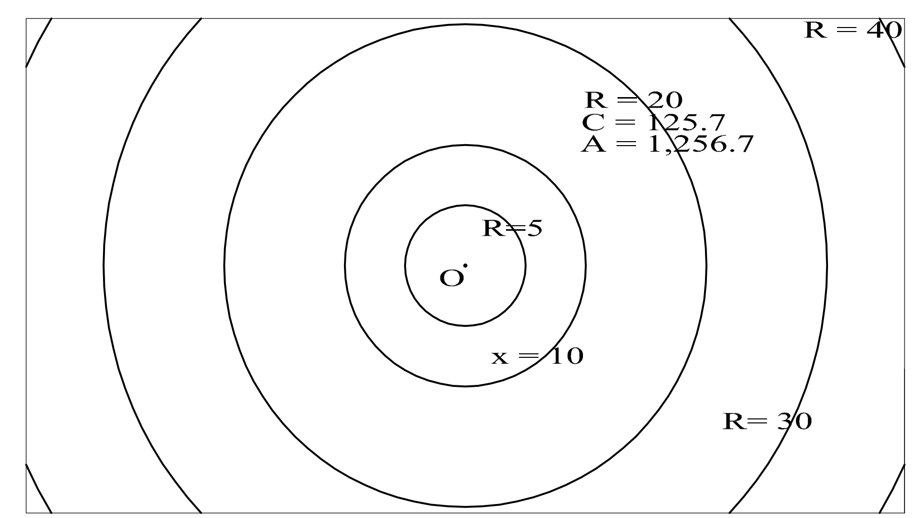
DRAWING 11: **Perspective views of a plane with *k* = 1,000.**



Type of Geometry | Constant factor | Distance to the Plane
Elliptic | k = 1,000 | z =100

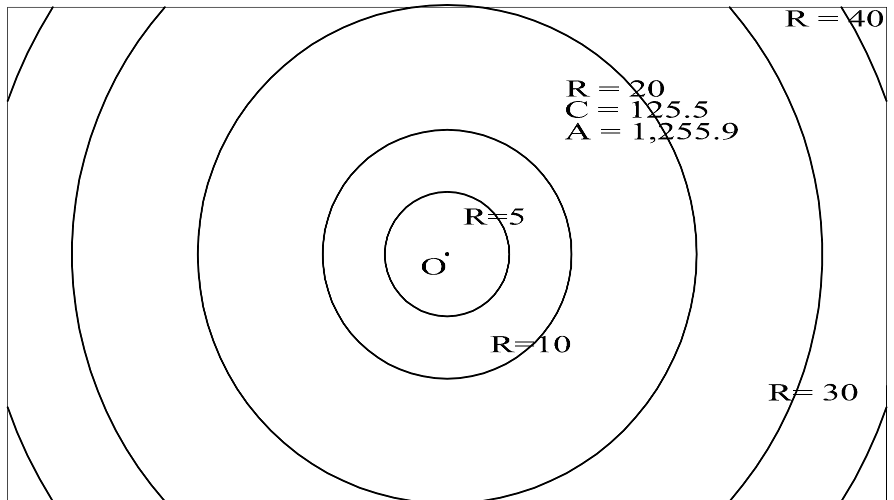


Type of Geometry | Distance to the Plane
Euclidean | z =100

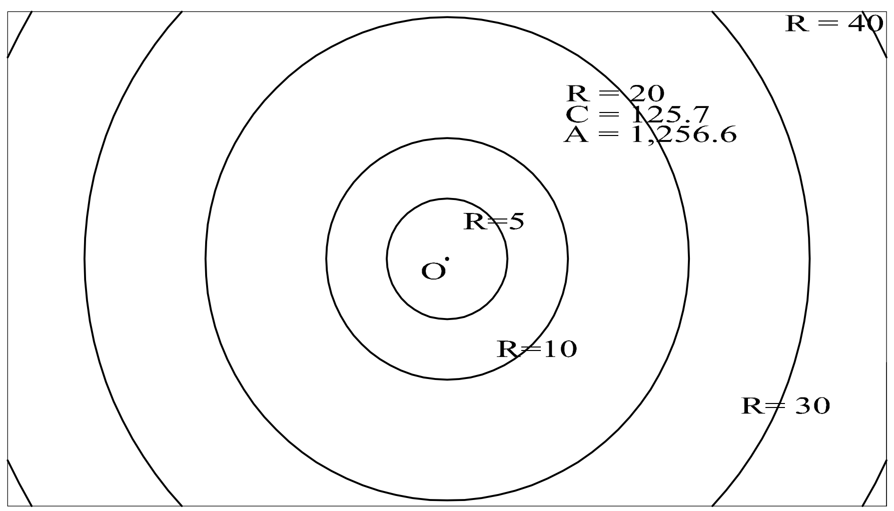


Type of Geometry | Constant factor | Distance to the Plane
Hyperbolic | k = 1,000 | z =100

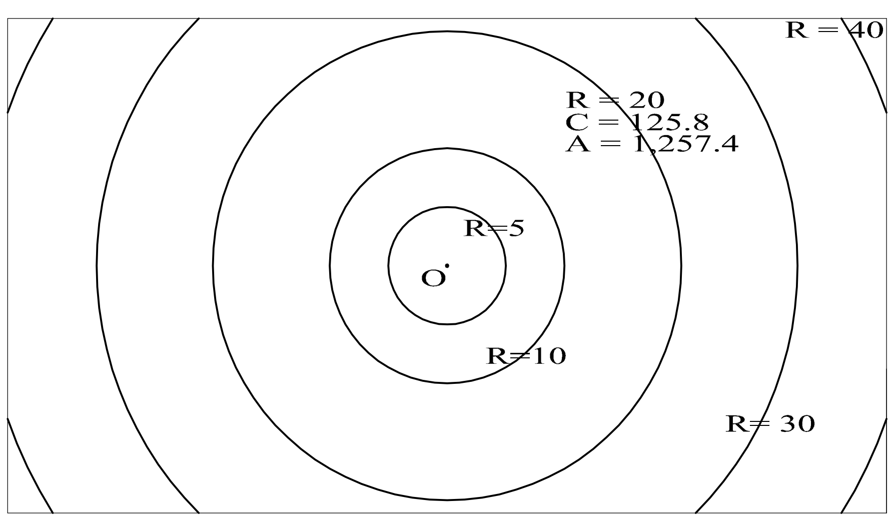
DRAWING 12: Perspective views of a plane with $k = 240$.



Type of Geometry Elliptic | Constant factor k = 240 | Distance to the Plane z =100

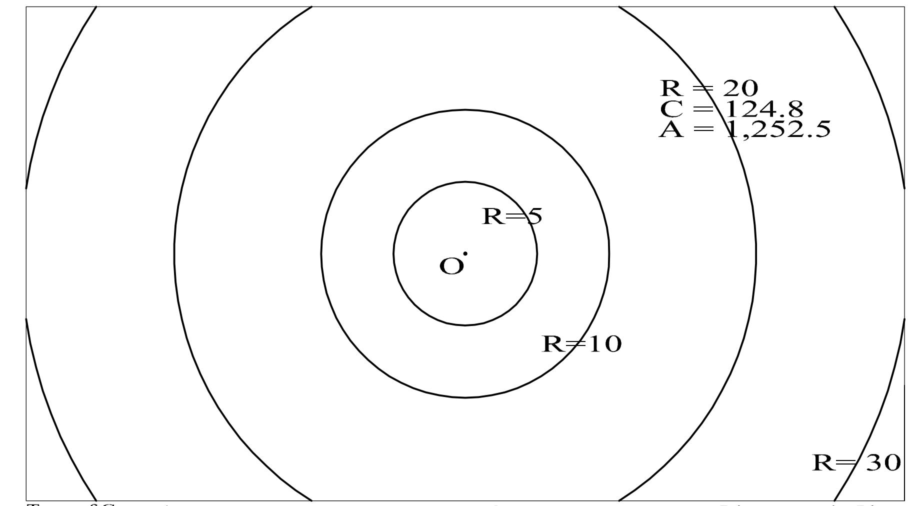


Type of Geometry Euclidean | Distance to the Plane z =100

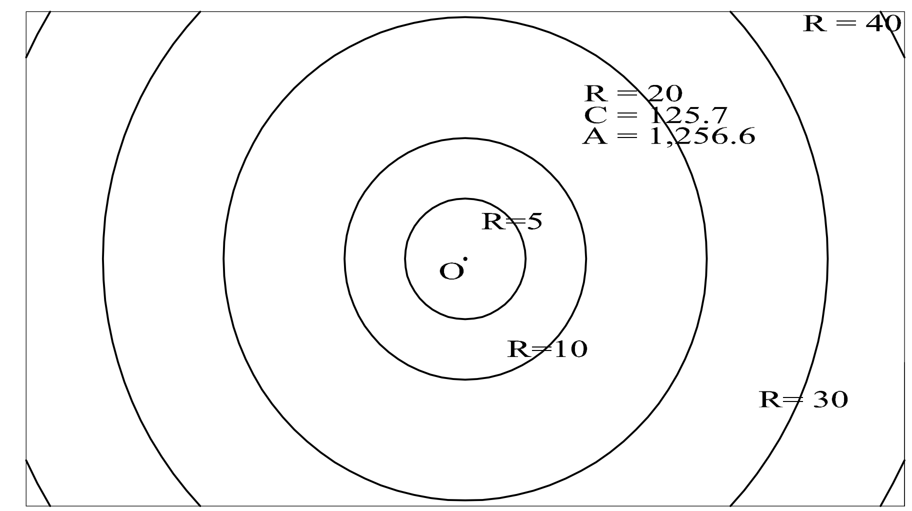


Type of Geometry Hyperbolic | Constant factor k = 240 | Distance to the Plane z =100

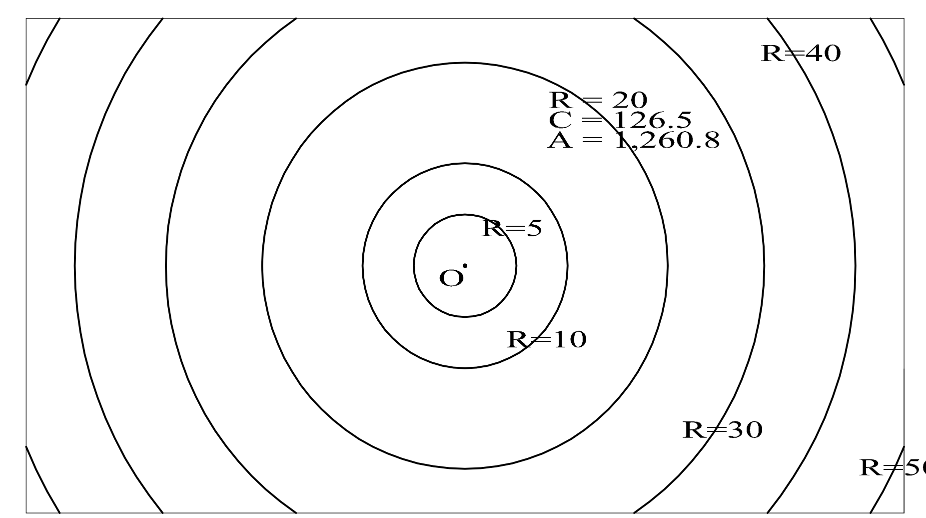
DRAWING 13: Perspective views of a plane with $k = 100$.



Type of Geometry Elliptic | Constant factor k = 100 | Distance to the Plane z =100

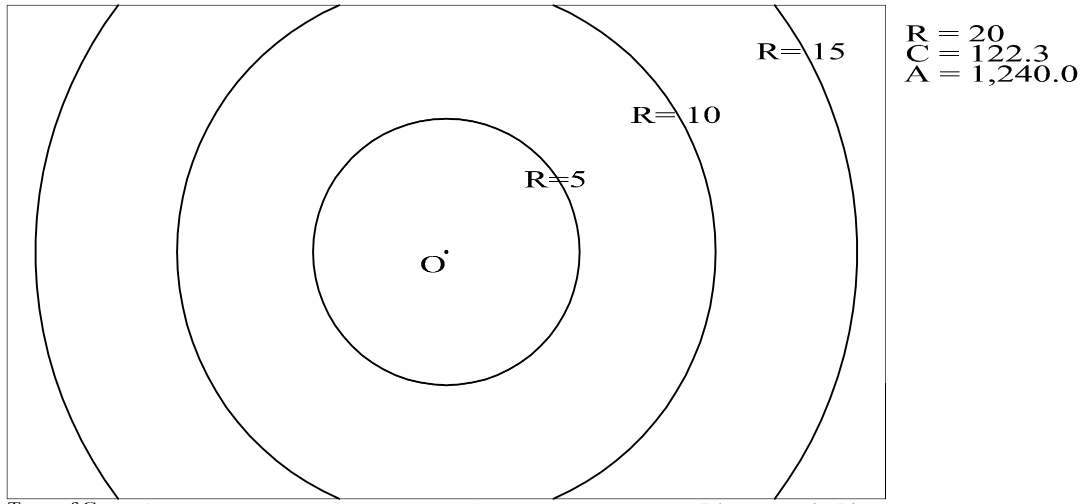


Type of Geometry Euclidean | Distance to the Plane z =100

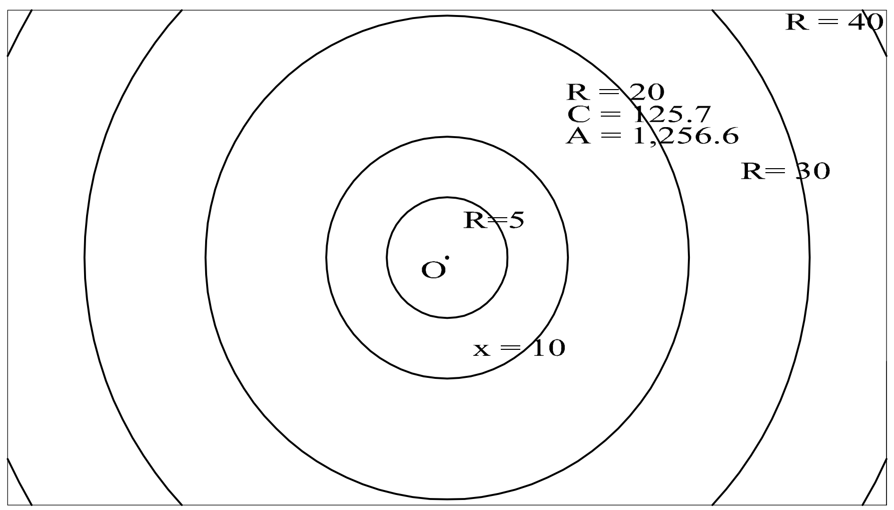


Type of Geometry Hyperbolic | Constant factor k = 100 | Distance to the Plane z =100

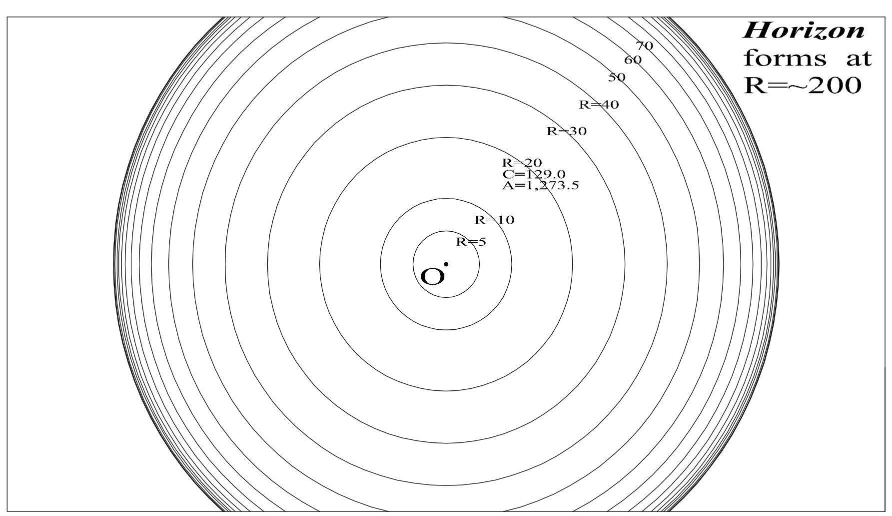
DRAWING 14: **Perspective views of a plane with $k = 50$.**



Type of Geometry **Elliptic** Constant factor **k = 50** Distance to the Plane **z =100**

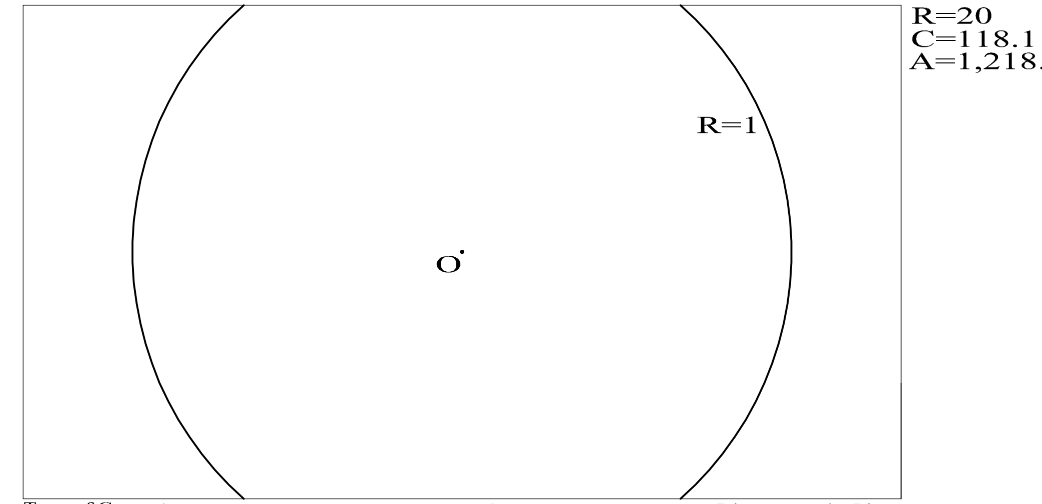


Type of Geometry **Euclidean** Distance to the Plane **z =100**

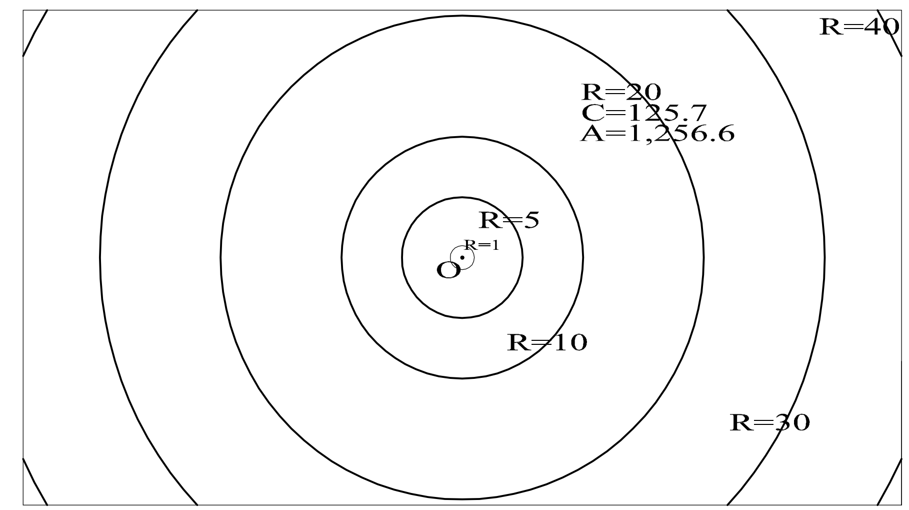


Type of Geometry **Hyperbolic** Constant factor **k = 50** Distance to the Plane **z =100**

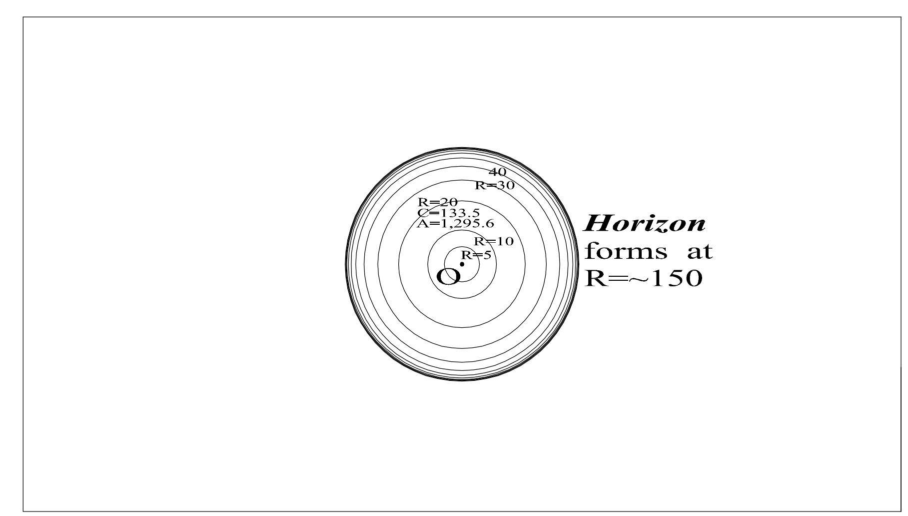
DRAWING 15: **Perspective views of a plane with $k = 33$.**



Type of Geometry **Elliptic** Constant factor **k = 33** Distance to the Plane **z =100**



Type of Geometry **Euclidean** Distance to the Plane **z =100**

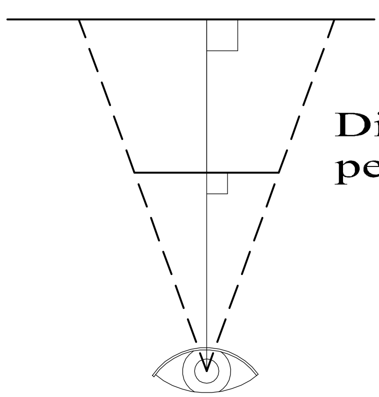


Type of Geometry **Hyperbolic** Constant factor **k = 33** Distance to the Plane **z =100**

With the "k factor" at 1,000, the Elliptic and Hyperbolic Geometries in *Drawing 11* are Approximately Euclidean. As the "k factor" decreases Non-Euclidean effects become apparent. The Elliptic space appears to be stretching out around our eye; the Hyperbolic space looks as if it were curving away from us. At k=50 our Perspective apparatus sees across the entire span of an infinite Hyperbolic plane.

Drawing 16 is a diagram of the actual measured distances from the Eye to various radius points on the plane. The paradox is that though those distances would tell us in Euclidean Geometry that the plane is curving, yet the Non-Euclidean planes remain perfectly flat. "Curving" is only an optical effect, and a somewhat misleading term. Better terminology might say that the Elliptic plane is ***converging*** toward our picture-plane and the Hyperbolic plane is ***diverging*** away from our picture-plane.

In terms of our original assumptions, we are seeing this:

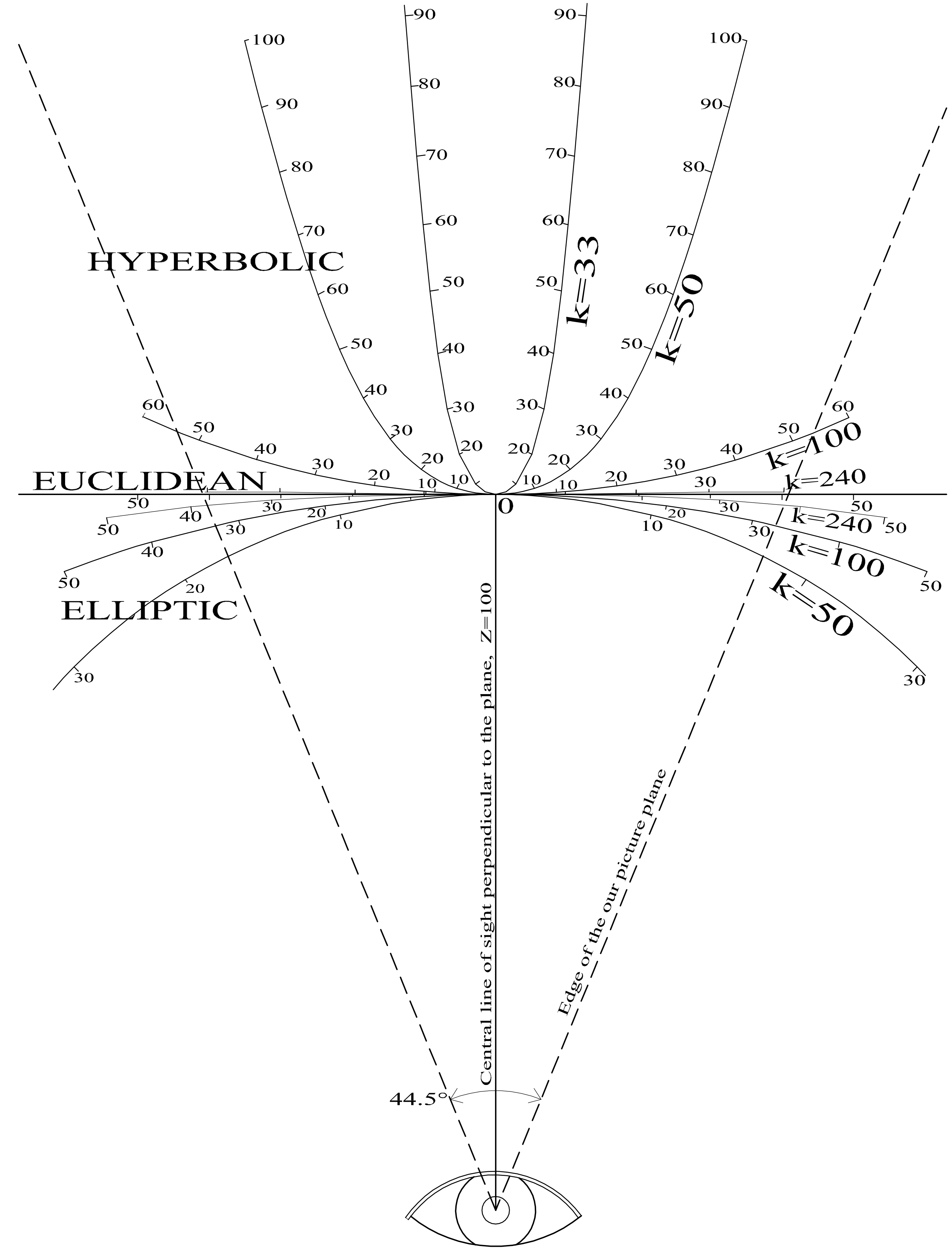


Distance between the 2 mutually perpendicular planes is:
- Converging (*Elliptic*),
- Remaining Equal (E*uclidean*), or
- Diverging (*Hyperbolic*)

The shapes of figures on a flat plane of Non-Euclidean Geometry appear to change as they slide around because what the Eye sees is similar to viewing portions of those figures turned at various angles in Euclidean space. The converging, and diverging, planes give a visual impression similar to Perspective "foreshortening" in our usual Euclidean views.

The central region of all our planes, the point where our line of sight is perpendicular, appears Approximately Euclidean, and angles appear true to their measure at point "O". But moving outward from that center-point, figures and distances appear distorted, as if we were rotating the figure in a Euclidean space.

DRAWING 16: **Diagram of the distances from Eye to plane**



The primary concept is that Non-Euclidean Geometries are changing the overall quantity of space. Elliptic Geometry is subtracting space, while Hyperbolic Geometry is adding space (compared to Euclidean).

At close range, the proportion of space subtracted or added is very small, and virtually immeasurable; but farther out, the quantities of volume being systematically subtracted from Elliptic space, or added to Hyperbolic space, become larger and larger.

DRAWING 17: **Diagram comparing spherical surface areas**

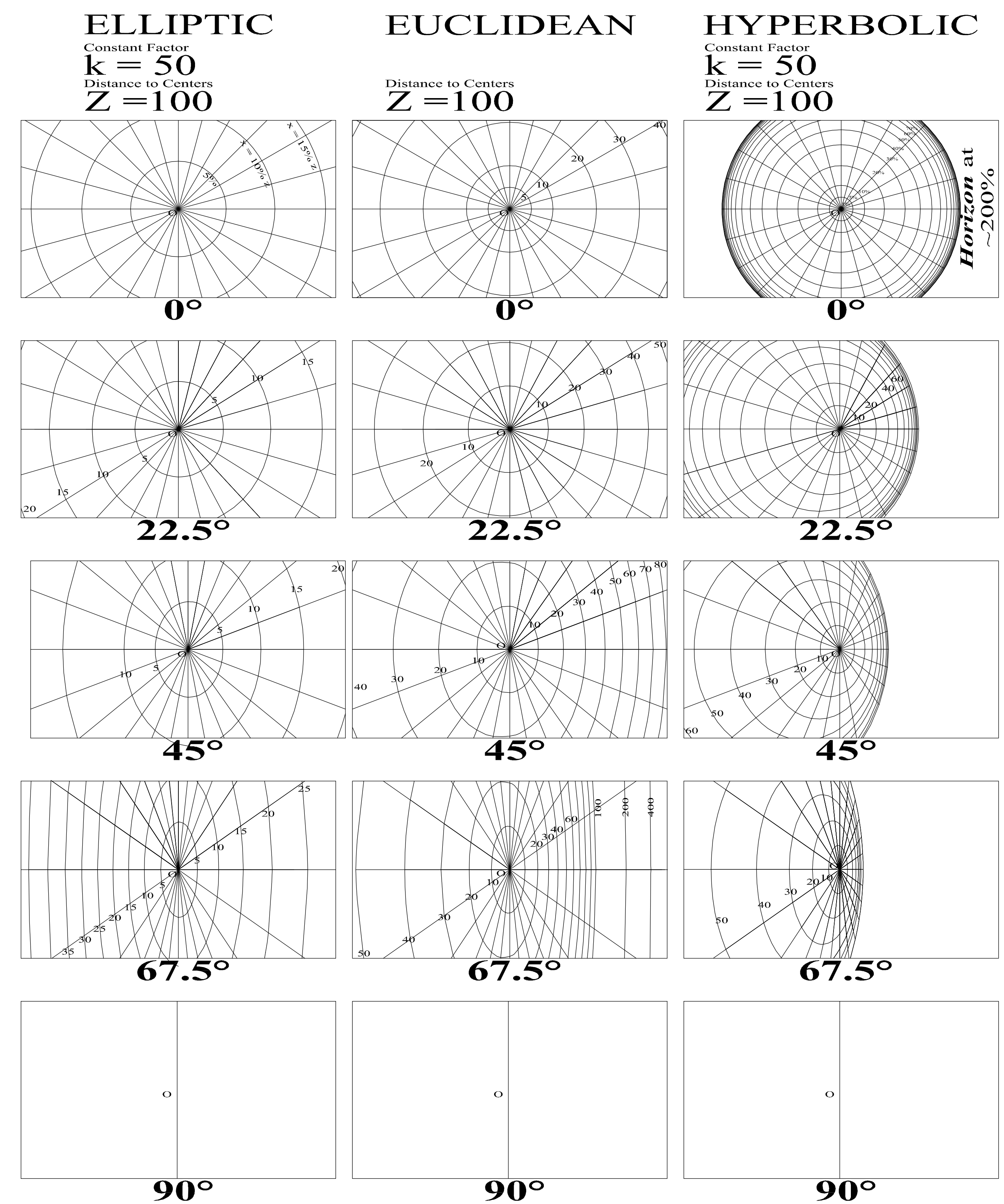
When we compare spherical surface areas, it is evident that the quantity of space is decreasing in Elliptic Geometry, and increasing in Hyperbolic .

ELLIPTIC **Spherical Surface Area** **k** = 100 **Percentage of Euclidean**	**RADIUS of the Sphere (Z =)**	**HYPERBOLIC** **Spherical Surface Area** **k** = 100 Percentage of Euclidean
100%	1	100%
92%	50	109%
71%	100	138%
60%	120	158
50%	140	185
39%	160	220
29%	180	267
21%	200	329

It is paradoxical to us that the planes of Non-Euclidean Geometry appear curved but are actually completely flat. It truly is a whole new type of space. To show you how flat it really is, in *Drawing 18* the plane is slowly rotated, turning about point "O", into an edge-on view.

In Hyperbolic Geometry we may call the circular outside rim the *Horizon*, the Hyperbolic plane infinitely receding. It is interesting to note that as the Hyperbolic plane rotates, that the *Horizon* stays intact. No portion of the planar surface ever "curves" outside our line of view. The entire expanse of the Hyperbolic plane (within the bounds of the picture frame) always remains visible.

DRAWING 18: **Perspective views of a rotating plane**
(Equally angled radial lines have been added to enhance visual clarity.)

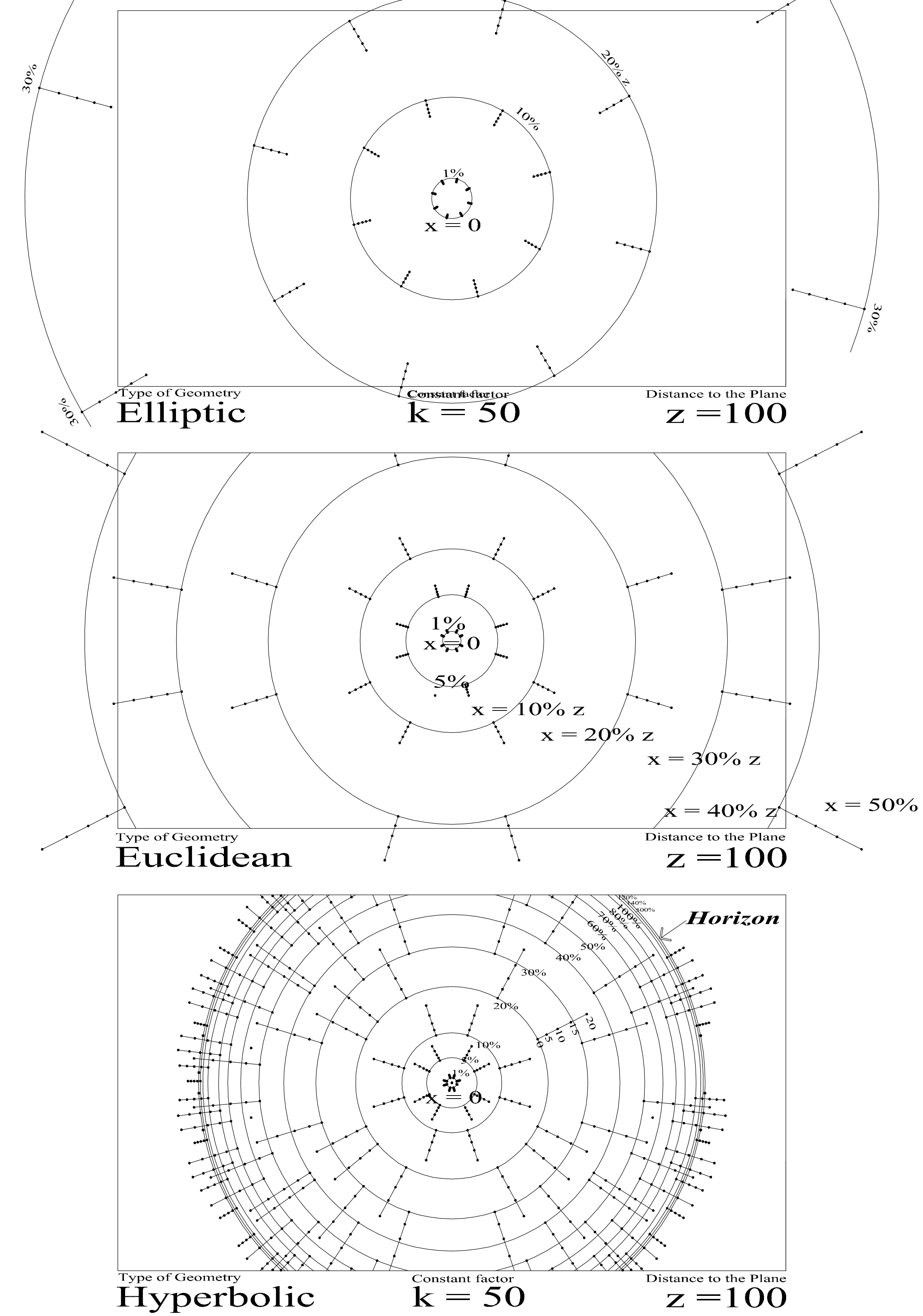


A view of a flat Hyperbolic plane with its circular Horizon serves as good symbolic representation for Hyperbolic Geometry. Straight lines appear straight. The central region is Approximately Euclidean in character and appearance, while as we move away from its center the outlying regions grows ever denser, the image of figures moving outward appearing ever smaller. Outer limits recede infinitely in every direction.

In *Drawing 19* I've taken the concentric circles of *Drawing 14* and scattered upon it equally long line segments set perpendicular to the plane (each perpendicular articulated with four equally spaced points). I think that these perpendicular elements help to visual the receding appearance of the *diverging* Hyperbolic plane and the concave illusion of the *converging* Elliptic plane. It helps to understand that the outer regions of the Hyperbolic Geometry's flat plane appear smaller because they are seen in foreshortened Perspective (rather than thinking that they are growing dimensionally smaller). Likewise for the Elliptic Geometry's flat plane, the concave aspect of its Perspective helps to understand the magnification of peripheral regions.

Chris: This is the "best" image set I've come up with (so far); and I would like to propose that you consider creating detailed renderings showing an imaginery aerial view of an urban towncape using this set. A square picture frame might work better. I would need to continue development of such a display -- to compute streets and building frames. A method will need to be invented that can illustrate the subtraction and addition of quantites of space (perhaps additional landscaping appearing in the Hyperbolic view, while conspicuous buildings disappeared from the Elliptic Perspective?) I envision something vaguely similar to Grant Wood's painting "*The Midnight Ride of Paul Revere"* -- a sort of Norman Rockwell small colonial-style townscape.

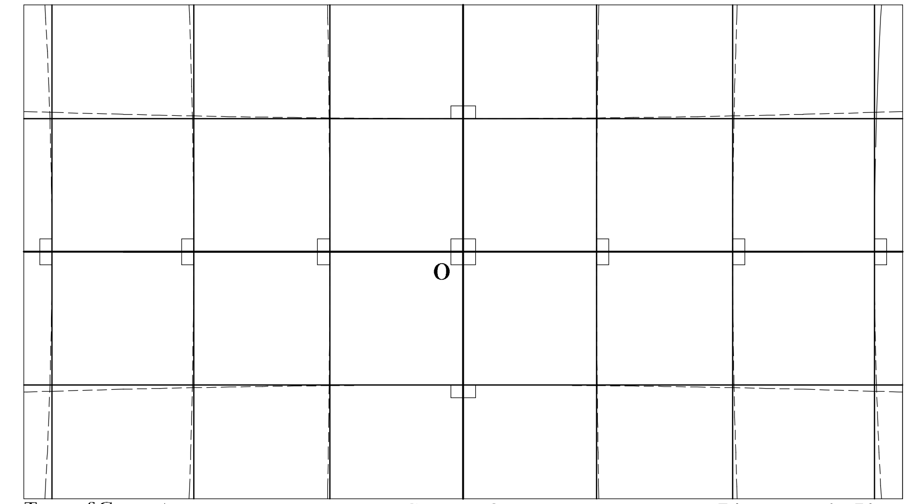
DRAWING 19: **Adding perpendiculars to the view of a plane**
(similar to *Drawing 14* on page 27)



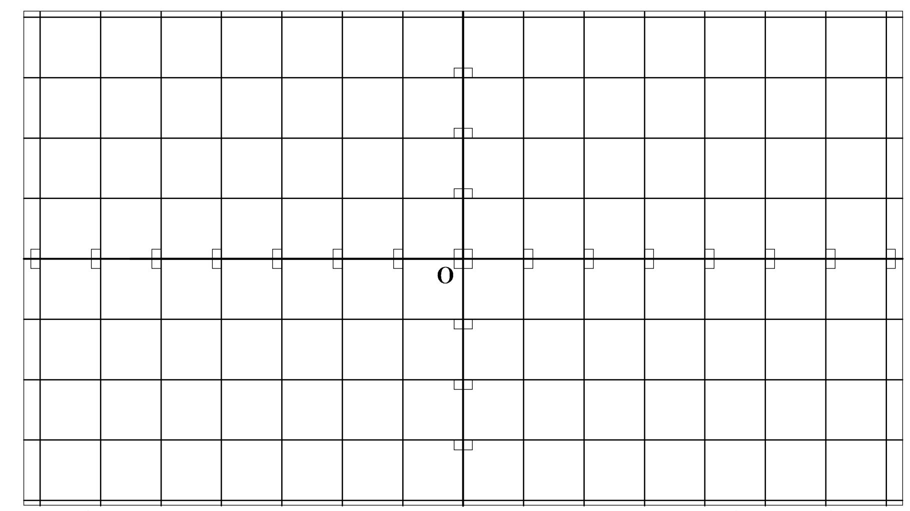
In *Drawing 20* the view of *Drawing 14* is repeated, again, but here the concentric circles are replaced by a pair of perpendicular *main axes* joined at point "O". Various secondary perpendicular lines are then cast off from each main axis with regular spacing. (Perpendiculars cast from any line passing through point "O" will appear as a right angle in the Perspective images of any of our three geometries.)

In *Drawing 20* there are also added Equidistant Curves, curves whose distance of separation constantly equals the distance between the perpendiculars at their construction points along the main axes. In Euclidean Geometry these sets of points form *parallel* straight lines and get superimposed on the adjacent perpendiculars; but in Non-Euclidean Geometry they form separate curves, shown here dashed. The Equidistant Curves illustrate the rate at which the perpendicular lines of Non-Euclidean Geometries *converge* or *diverge*.

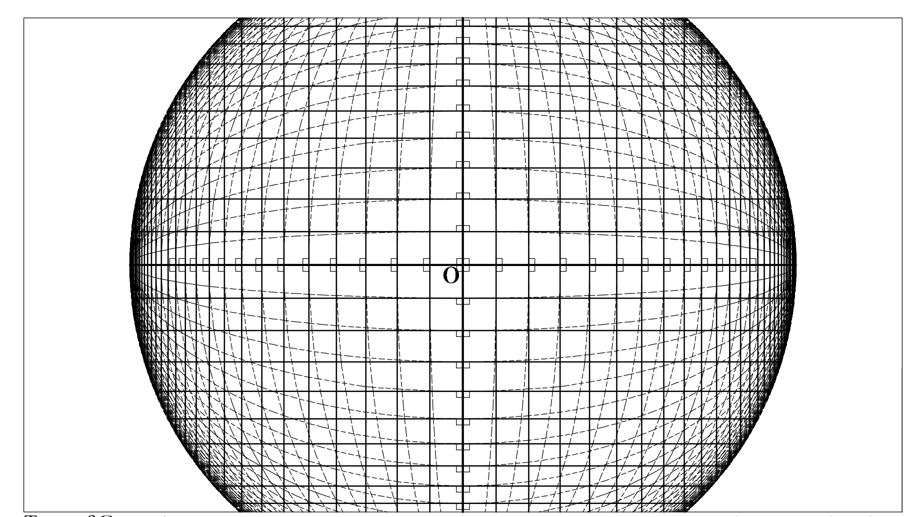
DRAWING 20: **Perspectives of a plane filled with perpendicular lines and Equidistant Curves**



Type of Geometry Elliptic — Constant factor k = 50 — Distance to the Plane z =100



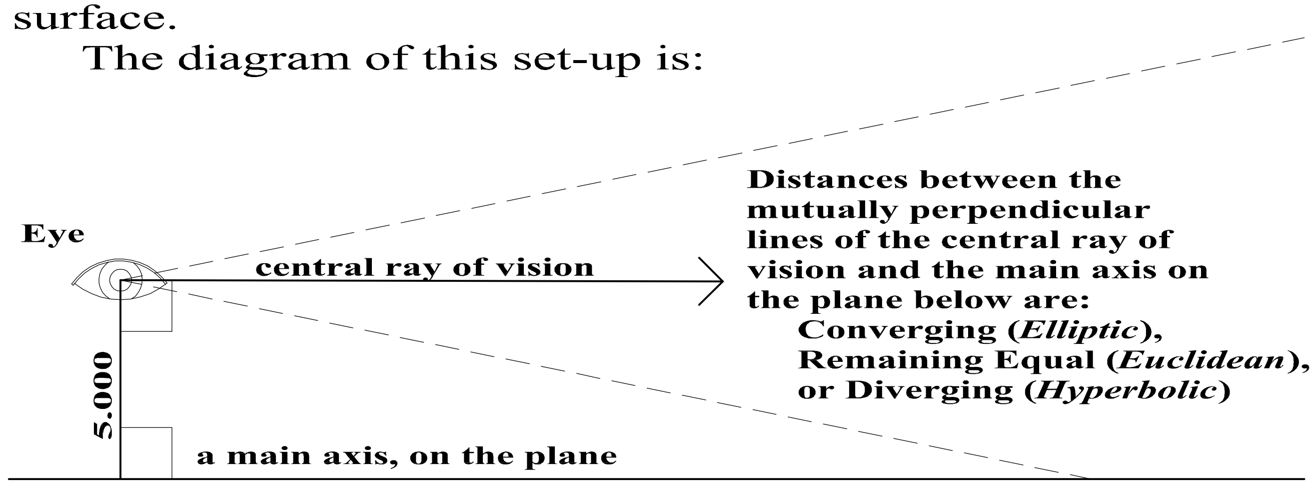
Type of Geometry Euclidean — Distance to the Plane z =100



Type of Geometry Hyperbolic — Constant factor k = 50 — Distance to the Plane z =100

As this chapter's final set of perspectives, *Drawing 21* lays down flat the previous plane, with its network of perpendiculars and Equidistant Curves (*Drawing 20*) and shows how it looks with our Eye looking out across its surface.

The diagram of this set-up is:



The human figures are added as perpendiculars and to show relative scale between the various views. They are purely diagrammic in detail. They violate our assumption of astromonical scale, but hopefully serve to animate these lifeless views. Perhaps they're odd interstellar nebulae.

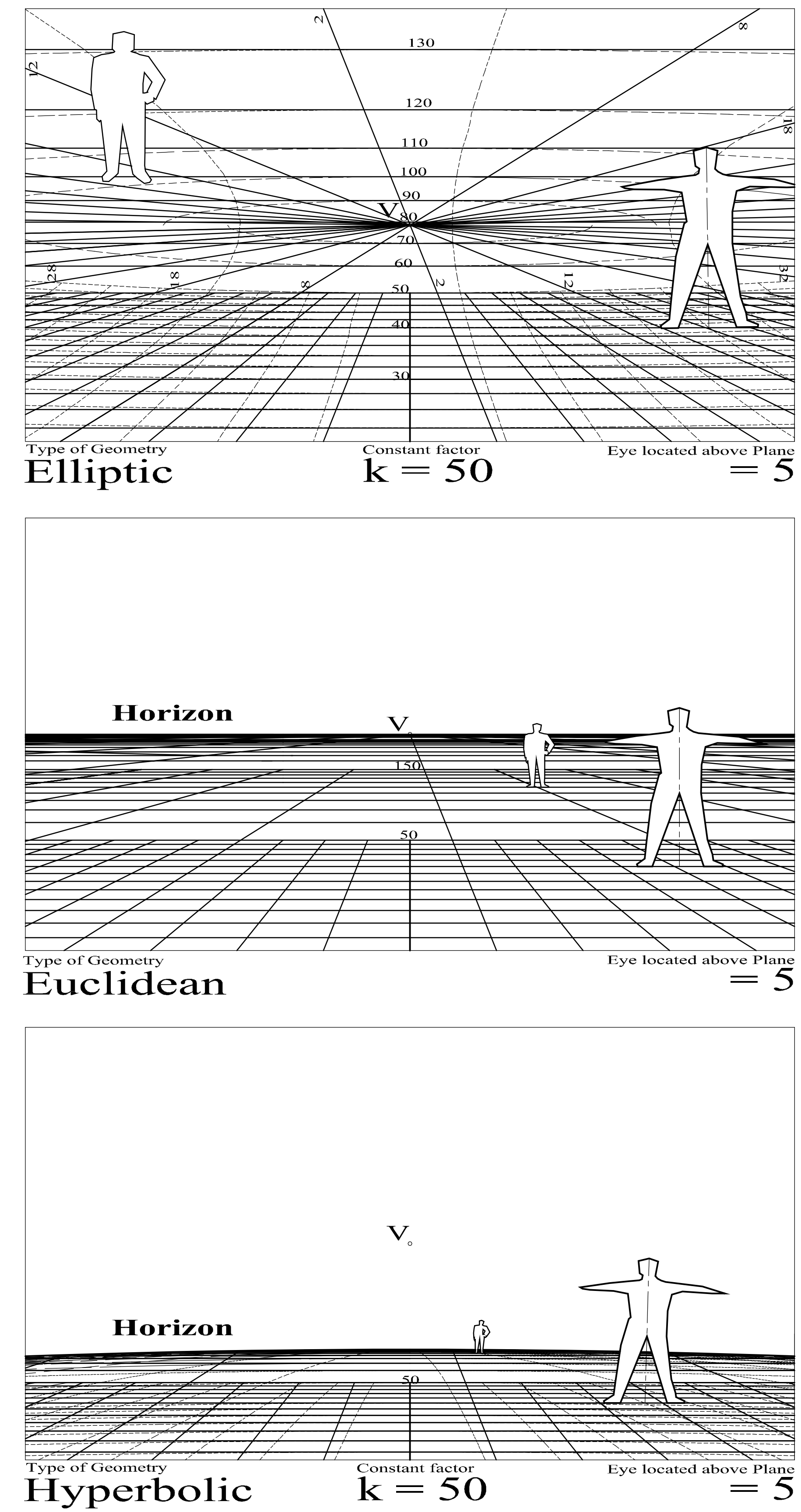
In all three geometries one set of perpendiculars all point toward the same vanishing point (V).

Both Eulcidean and Hyperbolic views have *Horizons*. The Hyperbolic plane forms its *Horizon* lower than the Euclidean view. Both the transverse perpendiculars and the human figures illustrate that distances appear to recede more quickly in Hyperbolic than in Euclidean space. No portion of the Hyperbolic plane's image disappears, its *Horizon* displays its infinite divergence.

From side to side, the Hyperbolic plane appears gently curved as it diverges away from the Eye's central sightline.

Discussion of the Elliptic Perspective starts on page 41.

DRAWING 21: **Looking out across a plane with perpendicular lines and Equidistant Curves**



This Elliptic Perspective was a struggle. Now that I see it, I could improve it. For example careful re-spacing of the perpendiculars could resolve the bits of Equidistant Curves near the center. But let me explain what I have.

A plane in Elliptic Geometry turns out to be similar to the surface of sphere, where a straight line is a *great-circle*, like Earth's equator, and the set of perpendiculars to that line are longitudinal meridians, converging at north and south *poles*. Perpendiculars cast from a single straight line in Elliptic Geometry will converge until they intersect at a *pole*. Elliptic Geometries with a single *pole* are possible (analagous to a Mobius Strip) but are not examined in this text. In this book, *Elliptic Geometry* means the *2-pole* version. In Elliptic Geometry every line is finite in length and boundless, its path always returning to its point of origin, a closed circuit.

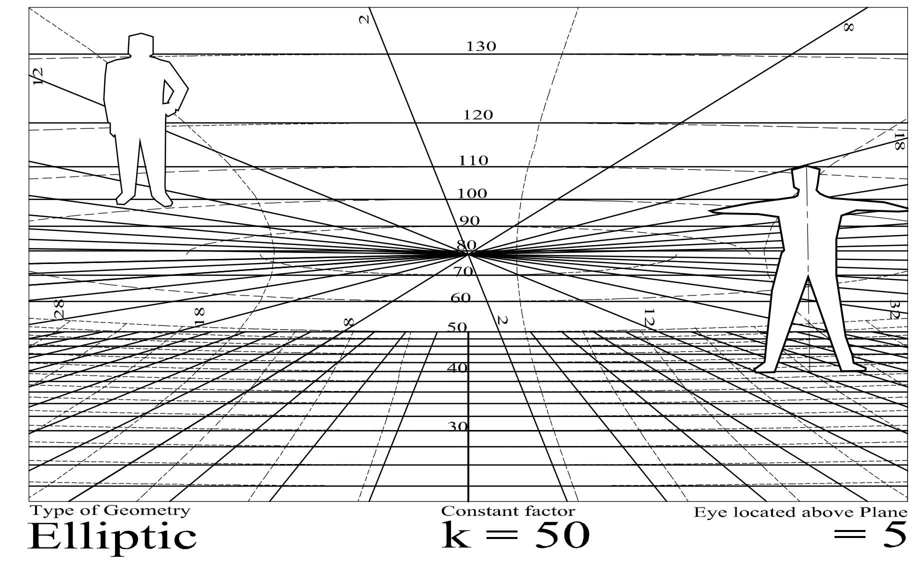
Here, in our *Drawing 21(a)*, all the perpendicular lines pointing forward intersect at the center of the picture. Their second *pole* would appear in the opposite direction, behind the Eye. Our second set of perpendiculars also converge, with one *pole* on the left and the other on the right. Those 4 views (front, back, left, and right) are all identical. The Eye's view directly overhead would be similar to *Drawing 20(a)*; its view down would be Approximately Euclidean.

Though difficult to sense in this Perspective, the image of this Elliptic plane side-to-side should appear to curl upward slightly, complimentarily opposite the slight side-to-side downward curl of the Hyperbolic image-- *Drawing 20(c)*.

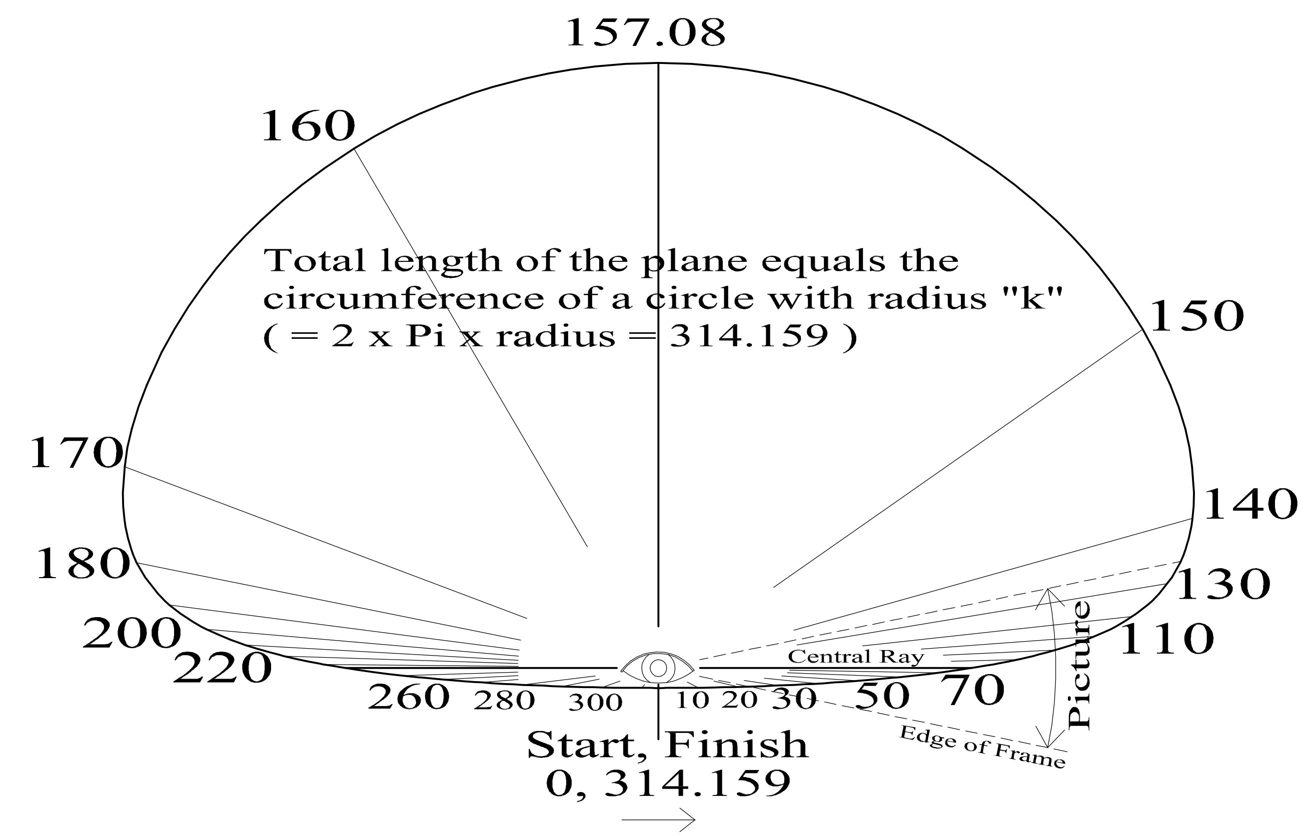
The most obvious feature of this view is the plane climbing as it recedes beyond the *pole*. The man at left is positioned along one of the lines crossing the *pole,* to illustrate that such lines continue on the picture's opposite side. I questioned whether that figure should be mirrored, or inverted, but concluded that such effects would depend on exactly how the man's construction had been specified.

Drawing 23 graphs the angles and distances of various points along one of this plane's main axes (with respect to the Eye). Paradoxically the plane is truly flat. In somewhat similar manner Elliptic lines (and planes) all appear to circuit around the Eye.

DRAWING 21(a): **(Repeated)**



DRAWING 22: **Diagram plotting angles and distances from the Eye to points along the central main axis of the plane viewed in *Drawing 20(a)*, above.**



If the finite lines of elliptic space return to their origins, would the Eye be able to see itself in *Drawing 22(c)*? My equations fail at the precise angle of the central line of sight, but when I set a line just slightly off center, at distance 0.001 below the Eye, I can plot its new pathway, graphed in *Drawing 23*. A similar path 0.001 above the Eye would be its mirror (with respect to the central line line of sight).

From such close approximations I extrapolate to conclude that, no matter how we turned the direction of the central line of sight, from this point the Eye's vision would always strike the plane, never return to see itself. As Perspective *Drawing 22(c)* confirms, in this particular case the central ray from the Eye intersects this plane exactly at the *pole* formed by the converging perpendiculars.

We may also predict the appearance of a small ship jetting through empty Elliptic space, heading away from our Eye precisely along its original central line of sight. Ahead, the Eye will see the rear of the ship receding but if it quickly turned, our Eye could also see the front of the ship much farther away, approaching from the opposite direction. Under proper conditions the Eye would witness a sudden faint flash filling the entire surrounding visual sphere, progressing rapidly from the direction the ship originally headed toward the opposite side of the sky. That flash would be the ship passing the midpoint of its circuit. Its ever receding stern might still be visible in the direction of its depature, even as an image of the ship's prow would be seen advancing from the opposite direction.

Under proper conditions, in an otherwise empty Elliptic space, the Eye would see the body of its own housing spread across the entire sphere of surrounding sky, its own back seen ahead, and its own top seen below.

A practical person should protest that these last examples violate our Perspective aparatus' original assumptions. By considering neither time-lags due to the speed of light nor Relativistic effects due to motion, the sizes implied for our subjects have become far too small to be deemed realistic.

In closing, I would like to mention that finite, boundless, self-returning lines are not necessarily unique to Elliptic Geometry. By additional postulates, Hyperbolic or Euclidean geometries may also acquire these, or similar, attributes. Such possibilities are subjects in Topology.

DRAWING 23: **A Diagram of the path of a ray similarly positioned "0.001" below the central line of sight, superimposed on the graph of an axis of the plane positioned at distance "5" below the Eye (*Drawing 22)*.**