Notes regarding the Hyperboloid Model

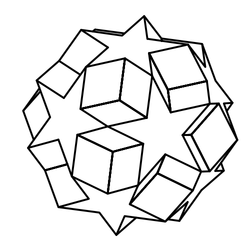
as a method of constructing visualizations of Hyperbolic Non-Euclidean Geometry

With four illustrations by
artist PETER STAMPFLI

CONTENTS

Caution -- different meanings of the word Hyperbolic:

The curved line called the Hyperbola, Hyperbolic Trigonometry, the Hyperboloid Model, and Hyperbolic Non-Euclidean Geometry are four separate subjects -- though related. Hyperbolic Trigonometry can be used to calculate Hyperbolic space but is not necessarily needed: the Hyperboloid Model can be used to visualize Hyerbolic space but is not required.



18th April 2022
Third Version of Third Draft

Jim Barnes (James David Barnes)
2006 Atlantic Street; Dallas, Texas 75208-3021; U.S.A.
(405) 525-3069; barnes444@sbcglobal.net; http://www.7Ladders.com

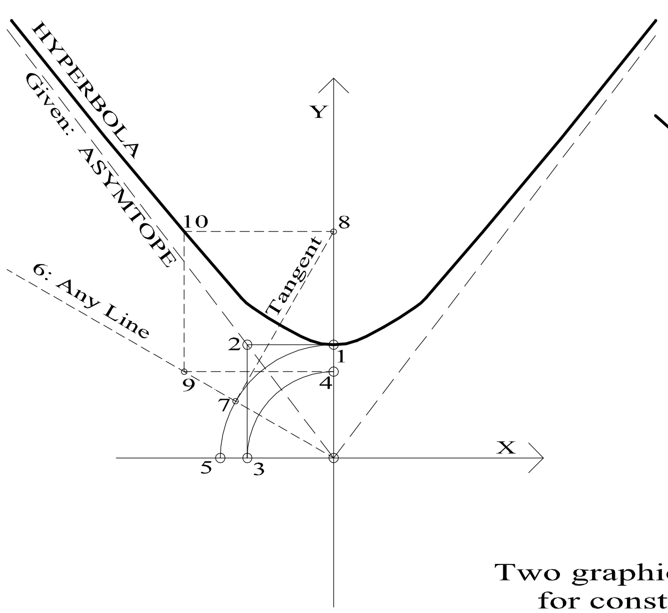
Most of today's visualizations of Hyperbolic Non-Euclidean Geometry use the Hyperboloid Model. The most typical such visualilzations are "the Poincare Model" and the "Klein Disk Model".

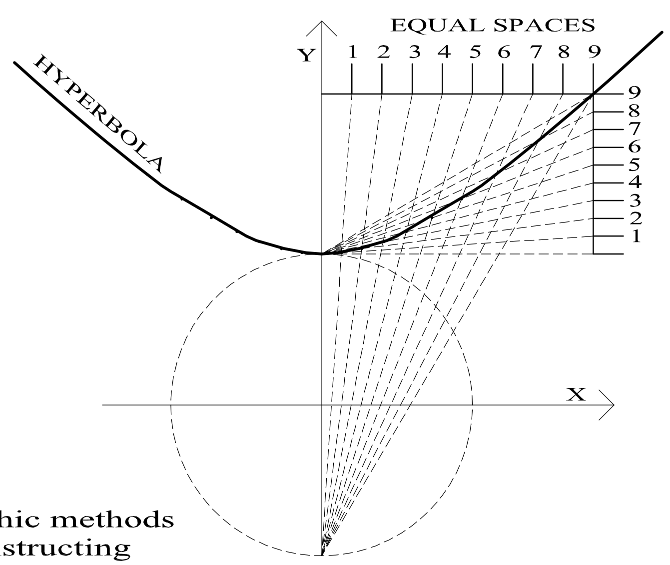
This presentation discusses the Hyperboloid Model as a basis for constructing visualizations -- from the point of view of a Perspective illustrator who is not especially skilled in mathematics.

THE HYPERBOLA

Hyperbolas as a class of curved lines were analyzed by geometor Apollonius of Perga more than two thousand years ago. The "unit hyperbola" was used to invent "hyperbolic trigonometry" around two hundred and fifty years ago -- about a century before the invention of Non-Euclidean Geometry.

Hyperbolic trigonometric functions were first developed to assist with natural logarithms -- mathematical "short-cuts" invented in the days before computers, when long tedious calculations were made by hand. Logarithms will not be discussed in this presentation.





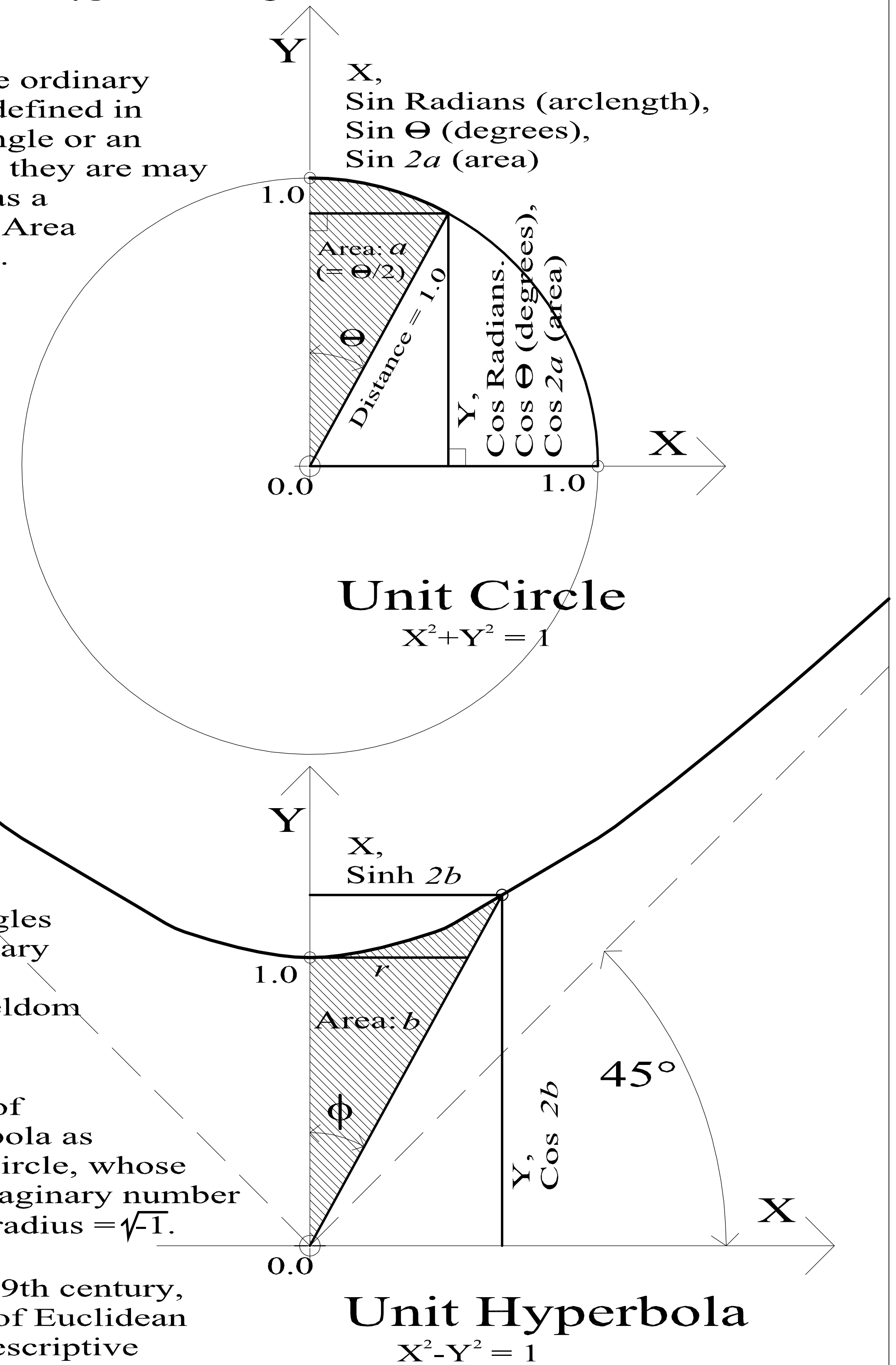
Two graphic methods
for constructing
HYPERBOLAS

via GRAPHIC STANDARDS,
Ramsey & Sleeper; 1970; p. 663

HYPERBOLIC TRIGONOMETRIC FUNCTIONS

A "unit circle" defines ordinary trigonmetric functions and a "unit hyerbola" defines hyperbolic trigonometric functions.

We usually see ordinary trigonometry defined in terms of an Angle or an Arclength, but they are may also be states as a function of an Area of a unit circle.



Hyperbolic Trigonomtry is described as functions of Area, with Angles being a secondary characteristic. Arclength is seldom mentioned.

I like to think of the unit hyperbola as an inside-out-circle, whose radius is an imaginary number "*i*", a circle of radius $=\sqrt{-1}$.

Until the late 19th century, visualizations of Euclidean Geometry ("Descriptive Geometry") had never used this Hyperboloid Model.

THE HYPERBOLOID MODEL

During the 19th Century it was discovered that existing hyperbolic trig functions fit to the new Non-Euclidean Hyperbolic Geometry.

Revolving the unit hyperbola into a unit hyperboloid, the following visualization model systems were developed:

1. The ***HYPERBOLOID MODEL*** (also known as the *Weierstrass Model*, the *Minkowski Model*, or the *Minkowski-Lorentz Model*).

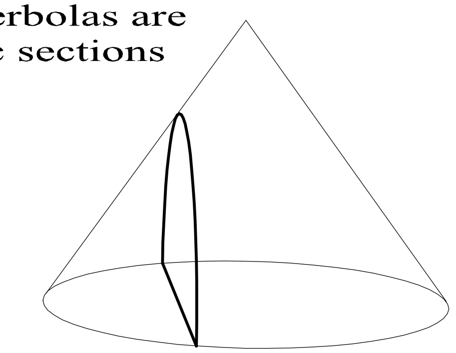
Images derived from it, by various projection methods, are:

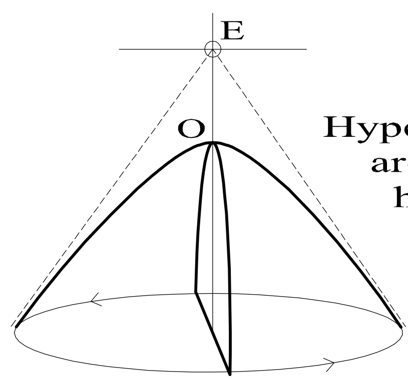
2. The ***KLEIN DISK MODEL*** (also known as the *Beltrami Model,* the *Beltrami-Klein Model,* the *Projective Model*, the *Cayley-Klein Model*, or *Central Projection);*
3. The ***POINCARE DISK MODEL*** (the *Conformal Disk Model);*
4. The ***POINCARE HALF-PLANE MODEL***; and
5. The ***GANS MODEL*** (*Orthogonal*, or *Orthographic Projection*)

To say that the Hyperboloid "represents" a plane surface in Non-Euclidean Hyperbolic Geometry gets to be a rather tricky and potentially confusing business. Such statement has the character of being an irrefutable definition -- but it's not exactly clear where the relationship is based. Certainly a flat plane in a Hyperbolic Geometry will always remains absolutely flat -- not curved.

The remainder of this presentation will study the Hyperboloid Model and how it serves as a projection device.

Hyperbolas are conic sections





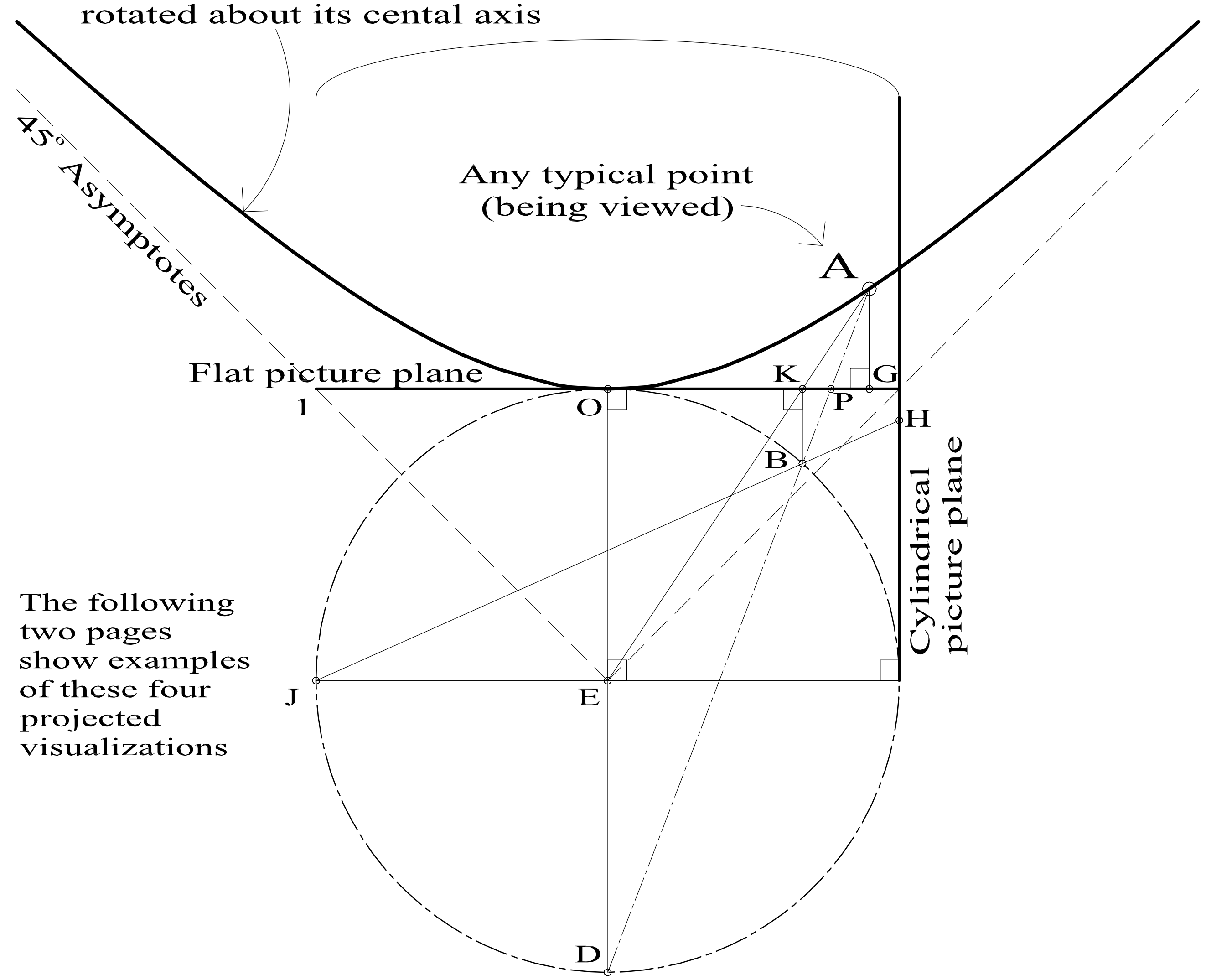
Hyperboloids are rotated hyerbolas

PREVIEW:

Where this is all headed is toward the conclusion that it might be better to say that the Hyperboloid Model represents the "k" factor -- the constant that regulates the densification of distance in Non-Euclidean space -- not so much the space itself. We proceed ...

FIVE MODELS for Visualizations of HYERBOLIC NON-EUCLIDEAN GEOMETRY:
A Euclidean "Section view", showing construction of all five models.

1. HYPERBOLOID MODEL:
a "Unit Hyperbola" (shown here) rotated about its cental axis



The following two pages show examples of these four projected visualizations

2. KLEIN MODEL: point "K" on the flat picture plane, projected from "E". ("gnomoic projection")

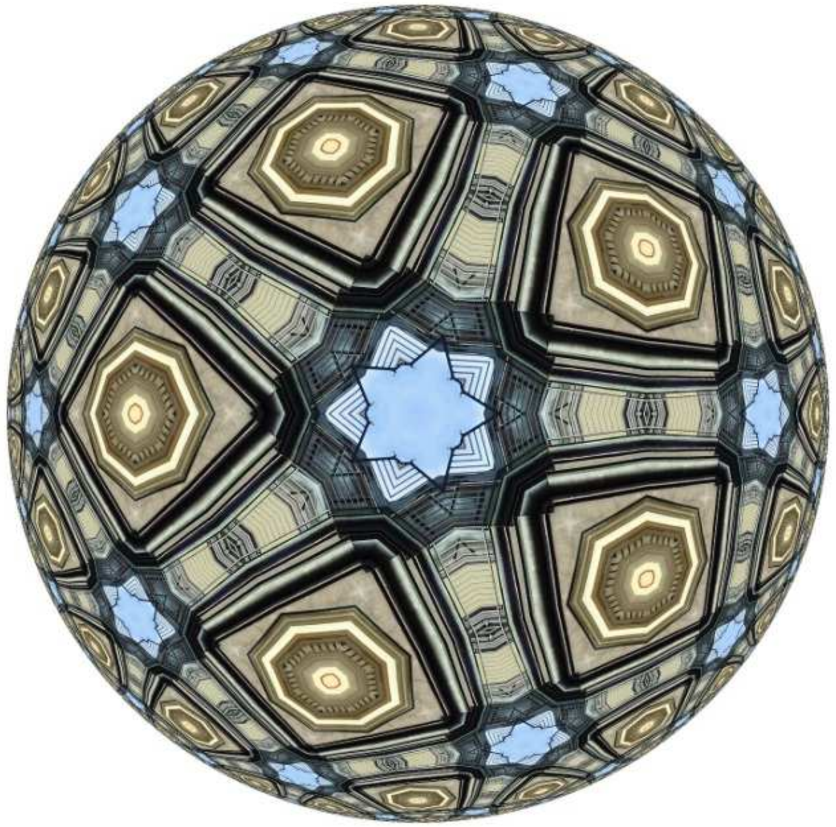
3. POINCARE MODEL: point "P", projected from "D". ("stereographic projection")

4. POINCARE HALF-PLANE MODEL: point "H" on a cylindrical image projected from "J", through "B" ("stereographic cylindrical projection)

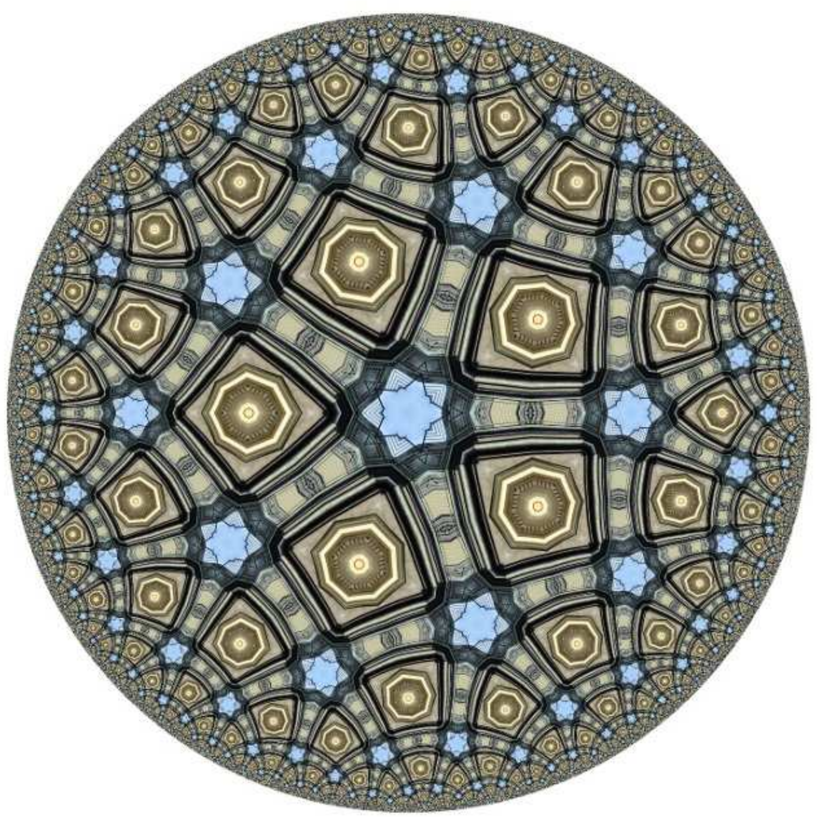
5. GANS MODEL: point "G" on the flat picture plane, projected orthogonally ("orthogonal projection")

Four visualizations of the Hyperboloid Model

by artist Peter Stampfli – used here by his permission (5th Dec. 2022), from his webpage:
https://geometricolor.wordpress.com/2018/11/04/various-projections-of-hyperbolic-kaleidoscopic-images/



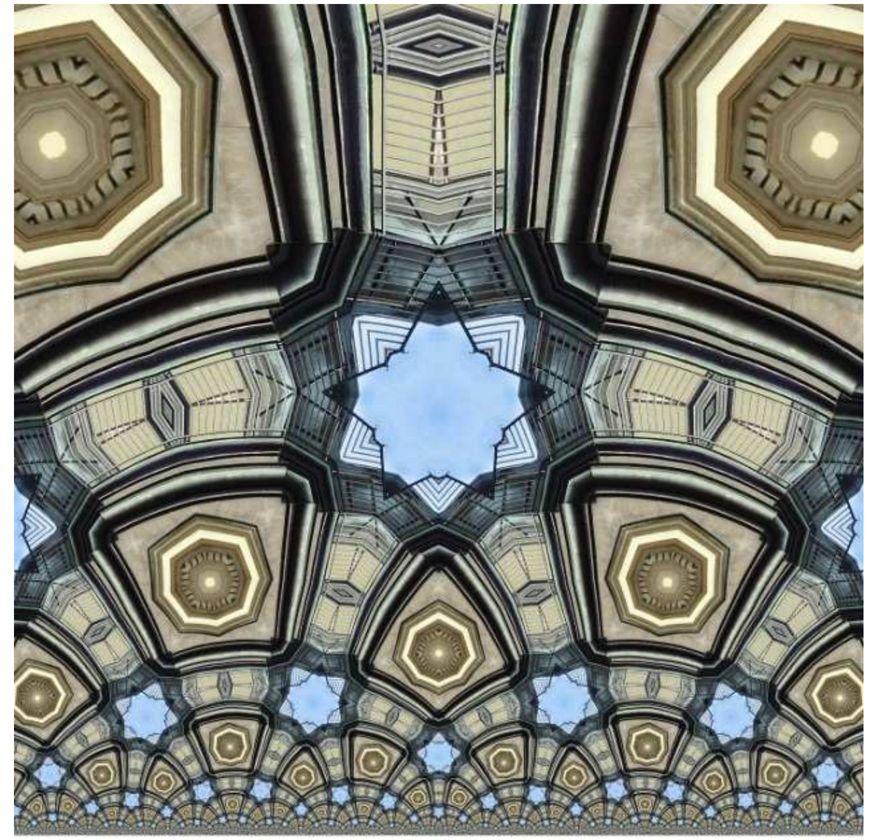
KLEIN MODEL



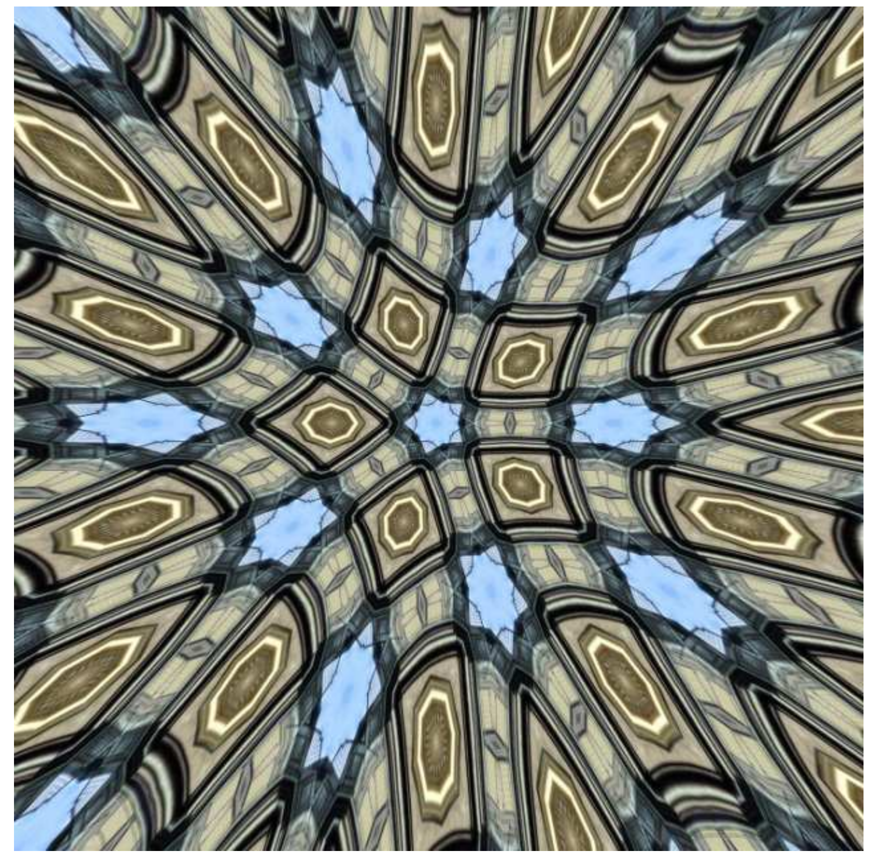
POINCARE MODEL

Create your own versions of these visualizations with this Peter Stampfli App --
http://geometricolor.ch/images/geometricolor/sphericalKaleidoscopeApp.html

POINCARE HALF-PLANE MODEL



GANS MODEL



DISTANCE DIAGRAMS DEFINED

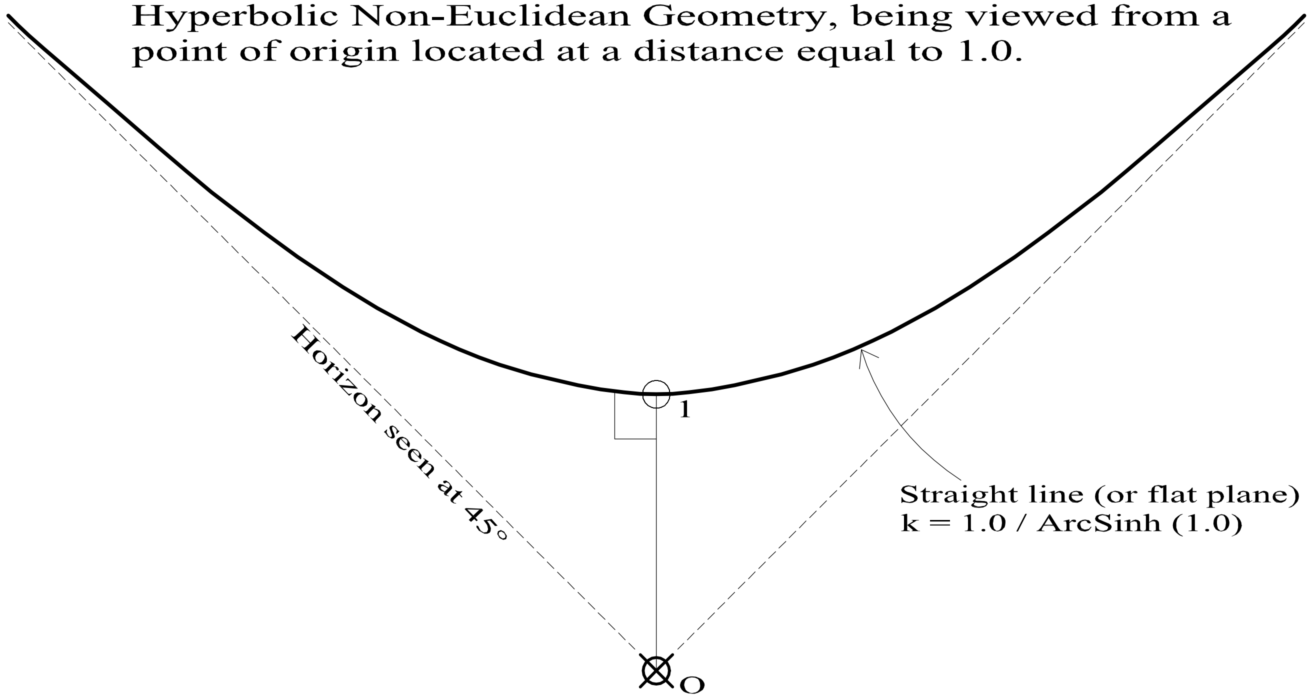
Before talking further about the Hyperboloid Model, let me define a measured graphic system I use -- I call it a "Distance Diagram".
Non-euclidean geometric straight-line figures have often been sketched using curved lines -- Distance Diagrams give such sketches a rigidous mathematical format.

This method of mapping figures of a Non-Euclidean space onto a flat Euclidean picture plane is simple. It starts at one (and only one) initial point, and then maps all the other Non-Euclidean points' distances at their angles with respect to the point of origin.

MATHEMATICAL DEFINITION OF A "DISTANCE DIAGRAM':
Starting from a single origin point (marked by symbol ⊗), let values R,ϕ,Θ of Non-Euclidean space become R,ϕ,Θ in Euclidean space.

This presentation will use only two dimensional Distance Diagrams.

Here is a Distance Diagram of a straight line (or flat plane) in Hyperbolic Non-Euclidean Geometry, being viewed from a point of origin located at a distance equal to 1.0.

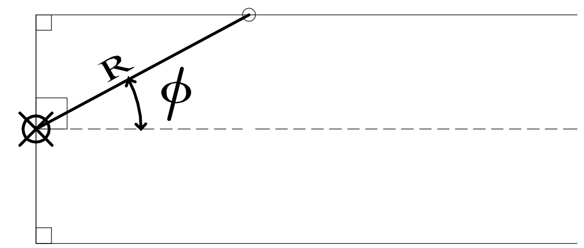


This looks like a unit hyperbola and the Hyperboloid Model, doesn't it? But it is not. The 'k' factor has been carefully chosen so that it is close, but it is not exactly the same figure.

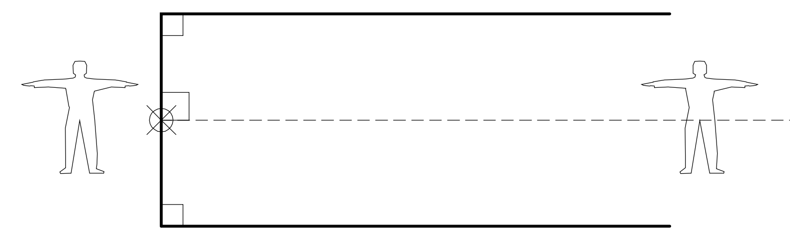
It turns out that such an exact match between the Hyperboloid Model and a Distance Diagram is impossible.

DISTANCE DIAGRAM

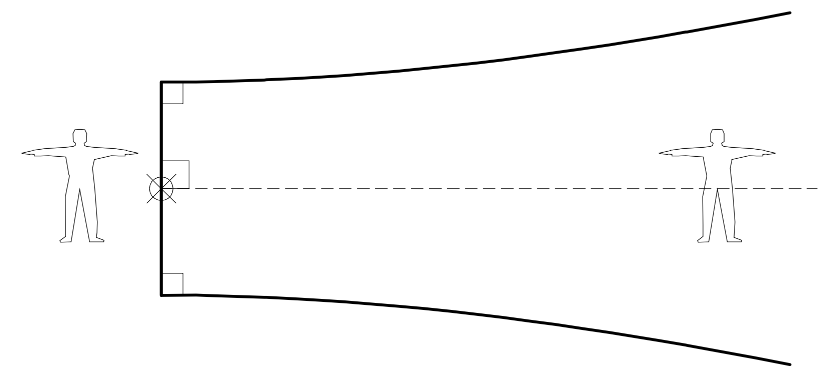
Start at mark ⊗ , the initial fixed reference origin,
every Point is plotted by transposing the Non-Euclidean lengths and angles into Euclidean values.



concept of figure

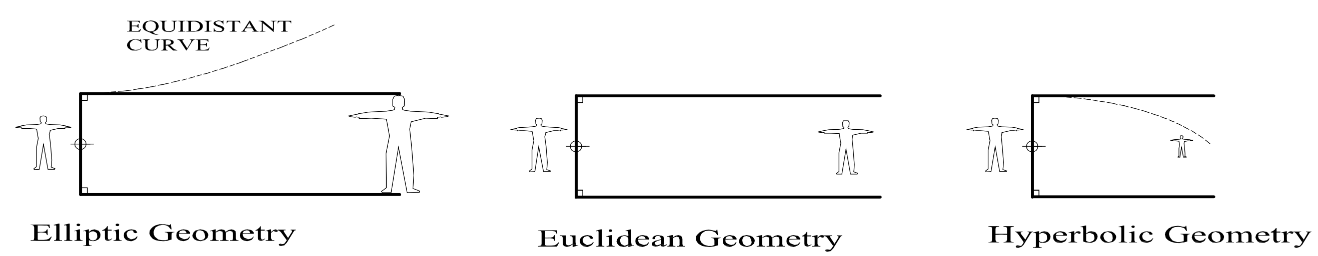


Distance Diagram of the Euclidean version of the figure

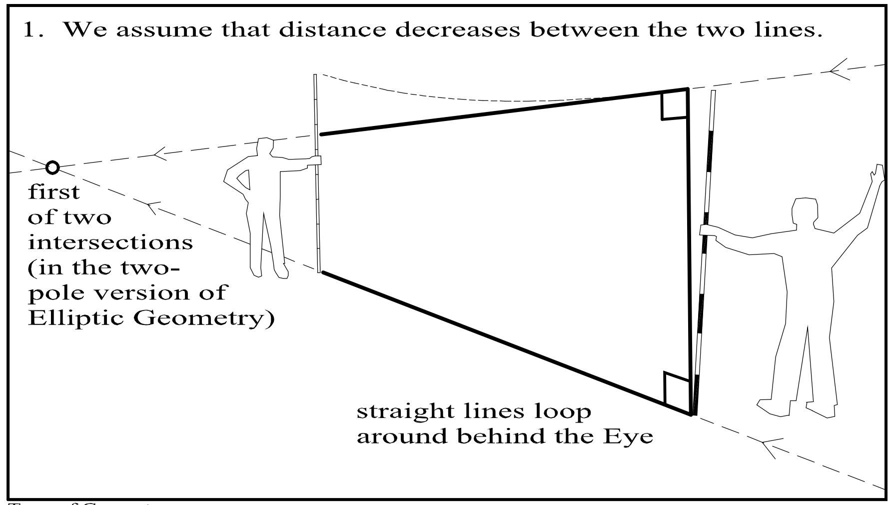


Distance Diagram of a Hyperbolic Geometry's version of the figure

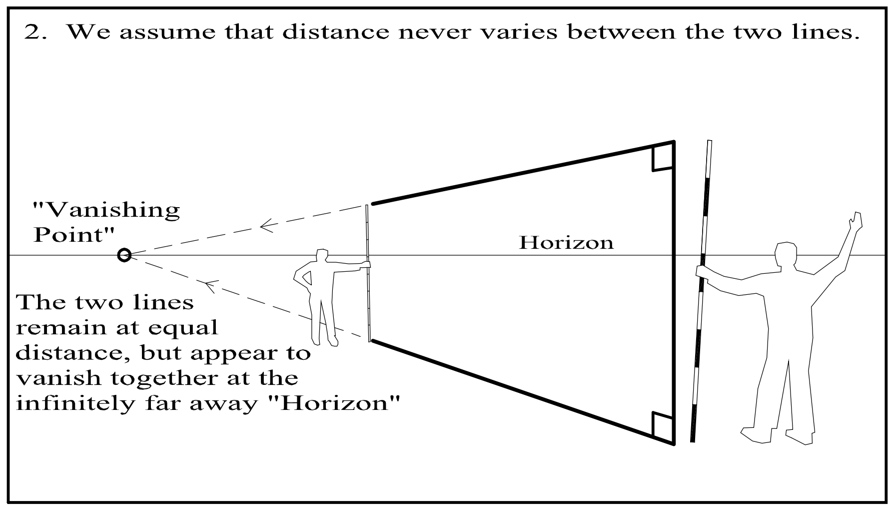
Comparative PERSPECTIVE views (where the Central Ray of Vision is perpendicular to the picture plane at the mark ⊕).



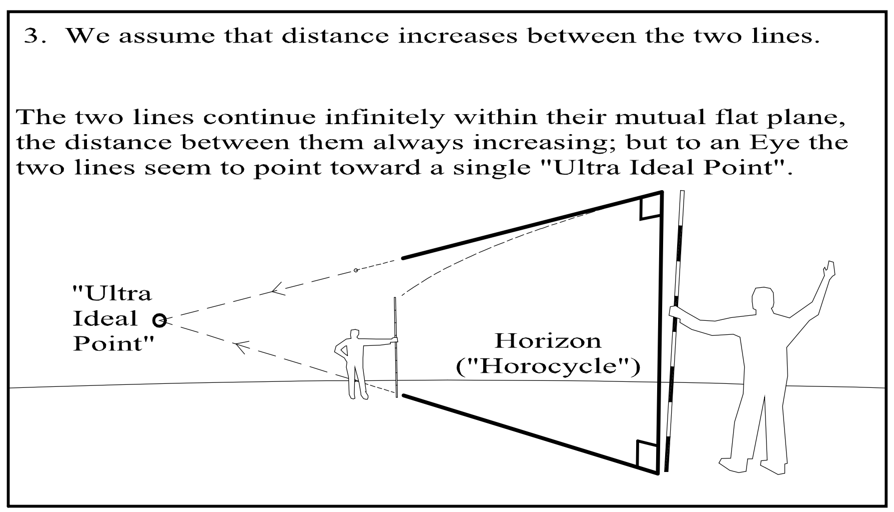
Three different assumptions create three different Geometries:
Assuming homogeneous ("isotropic") space, 2 co-planar lines are set perpendicular to a vertical. An Eye views 3 different possibilities.



Type of Geometry
Elliptic   Constant factor **k = 50**



Type of Geometry
Euclidean



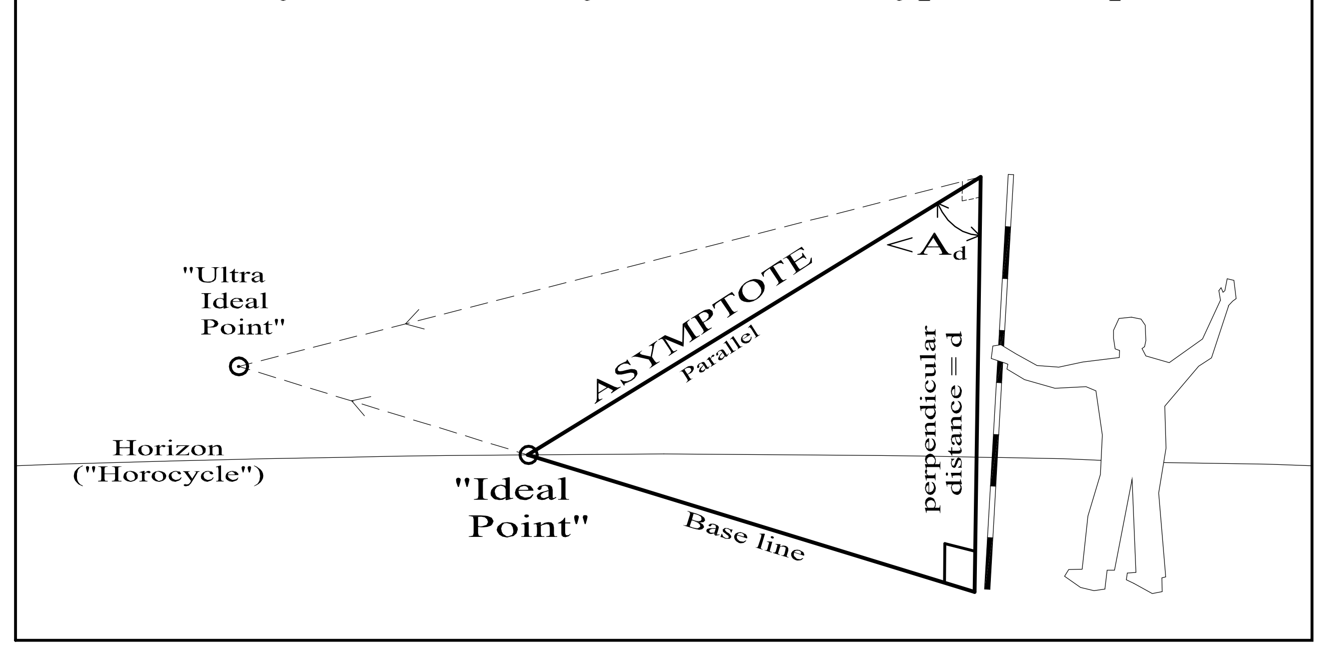
Type of Geometry
Hyperbolic   Constant factor **k = 50**

THE LAW OF ASYMPTOTES

"PARALLEL" was the unfortunate name given in earliest pioneering, which now usually contradicts more prevalent Euclidean meanings.

Every ASYMPTOTE and its base line are co-planar. They endlessly proceed to approach nearer together (in one direction) without ever actually intersecting.

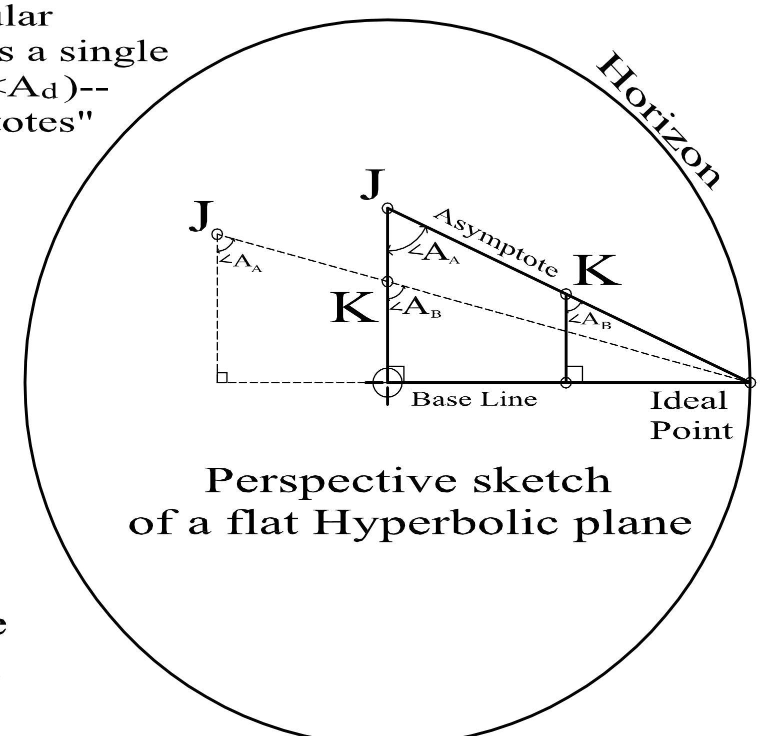
The Asymptotic Angle ($<A_d$) at an associated distance ("d") is consistently the same everywhere in the Hyperbolic space.

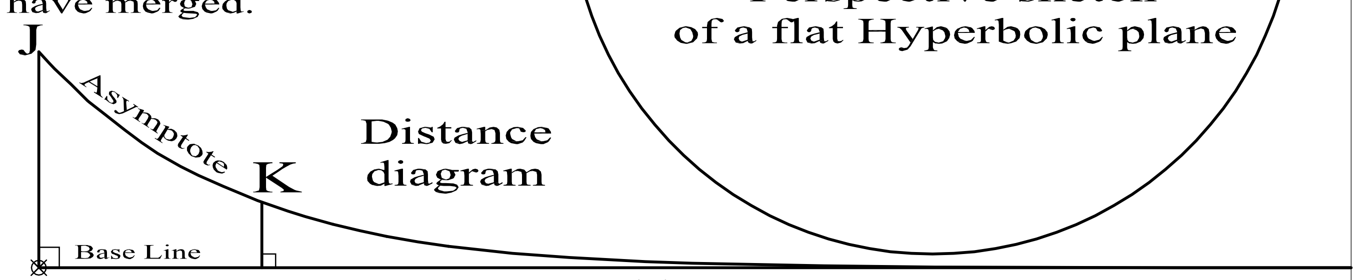


Type of Geometry
Hyperbolic   Constant factor **k = 50**

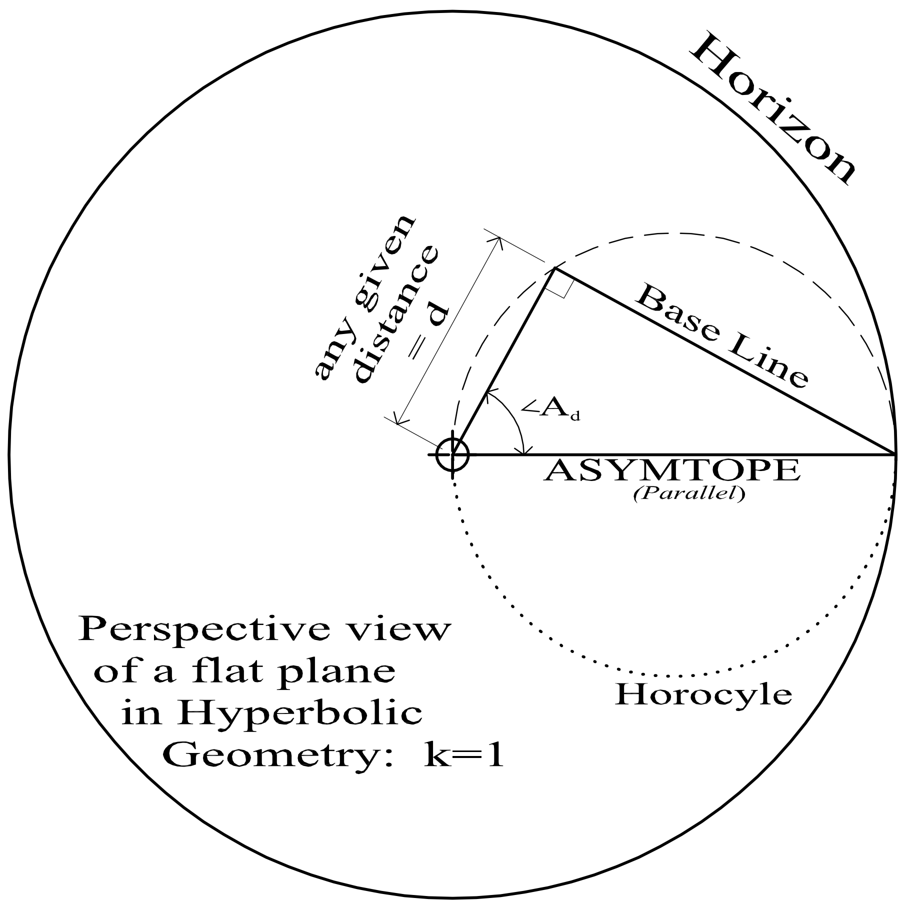
For each possible perdendicular vertical distance ("d") there is a single unique "asymptotic angle" ($<A_d$)-- making this "Law of Asymptotes" an important tool in deriving trigonometric forumlae for the Hyperbolic Geometry.

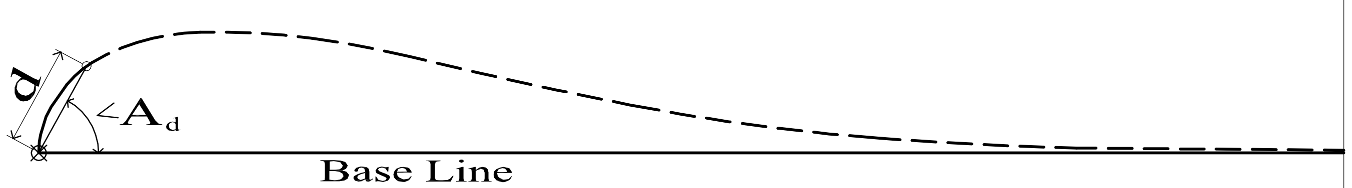
The slope of an Asymptote flattens into its base line as we shift our view toward the horizon. The two lines get ever closer; until, at infinity, they might be thought to have merged.





View of every possible distance "d" and every possible angle of *parallel* , with the angles of *parallel* set at the center of view so the angle appears in true proportion.



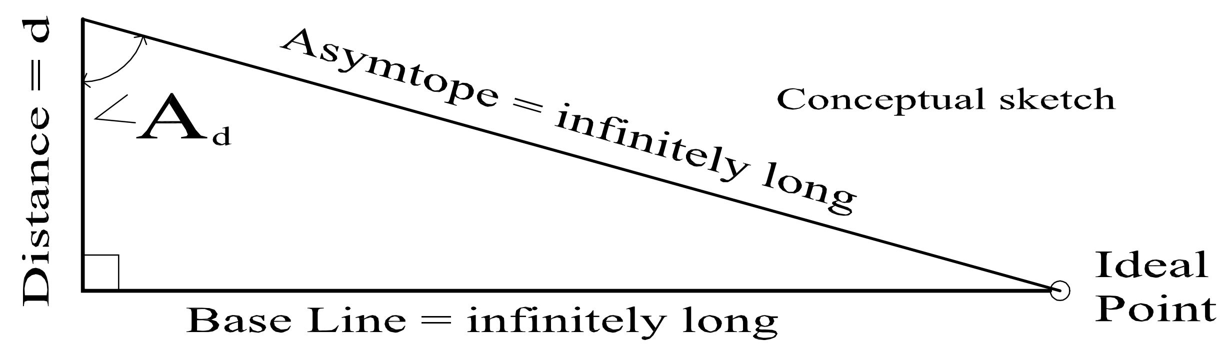


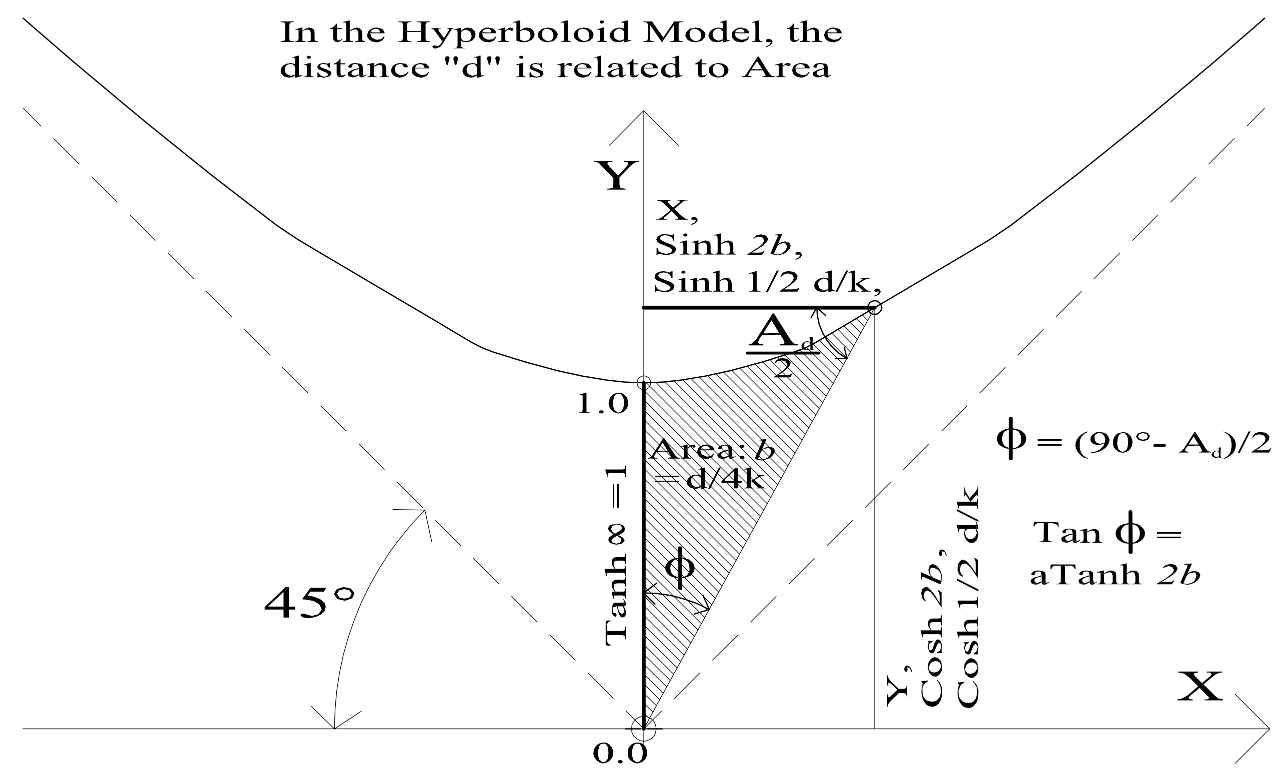
Distance Diagram

THE LAW OF ASYMTOPIC LINES SEEN IN THE HYPERBOLOID MODEL

Each vertical distance has a unique asymtopic angle.
Each asymtopic angle has a unique vertical distance.

Asymtopic Angles range from "approaching 0°", all the way up to "approaching 90°".
Vertical distances range from zero to infinity.





Hyperboloid Model

In the Law of Asymtopes, when k and d are both simultaneously multiplied by the same factor (any constant number), then the Asymtopic Angle remains unchanged.

CURVATURE -- THE "k" FACTOR

The "k" determines how quickly the density of distance between our two mutually perpendicular lines will enlarge or diminish in Non-Euclidean Geometries.

This essay assumes that the geometry under consideration has the same characteristic of ***curvature*** everywhere -- that the space is homogeneous ("istropic").

When values of "k" are very large, Hyperbolic and Elliptic spaces become approximately Euclidean. As "k" values become smaller, the Hyperbolic spaces become denser and the Elliptic spaces become less dense.

Non-Euclidean Geometry is sometimes said to be "curved space" ("warped space"), but nothing is actually being curved -- straight lines remain straight -- flat planes remain flat. The "k" factor is called "*curvature*" because it has the property of being the radius of a sphere -- the "degree of curvature" expresses the rate at which the sphere's surface is bending.

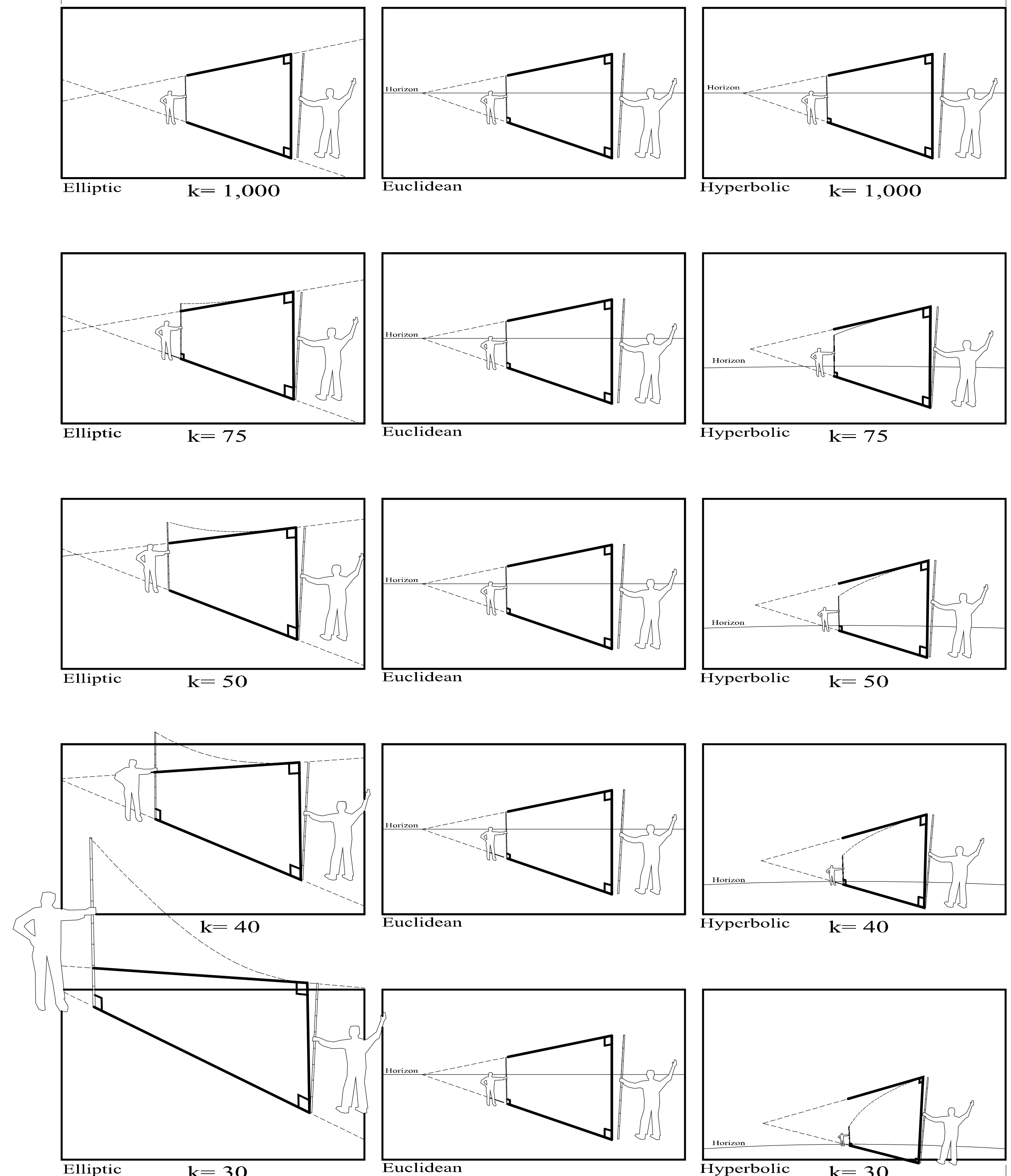
There is no "k" factor directly expressed in the Hyperboloid Model.

In Elliptic Geometry the radius "k" is a real number while in Hyperbolic Geometry "k" is an imaginary number. (The sphere of "k" turns "inside-out" in Hyperbolic spaces.)

My clearest mental image of the Hyperboloid Model is that it represents the archetypical "unit" value of "k"=$\sqrt{-1}$.

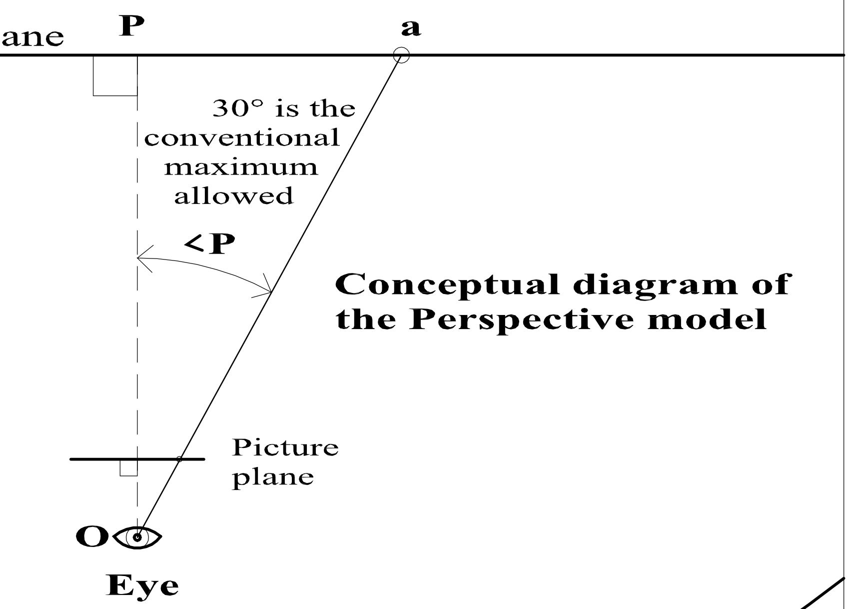
One of the "tricks" I discovered in trying to understand the Hyperboloid Model was remembering that a circle in Hyberbolic Geometry has a different radius/circumference ratio than in Euclidean Geometry -- the circumference of a circle in Hyperbolic Geometry = 2 (Pi) k (Sinh (radius/k)).

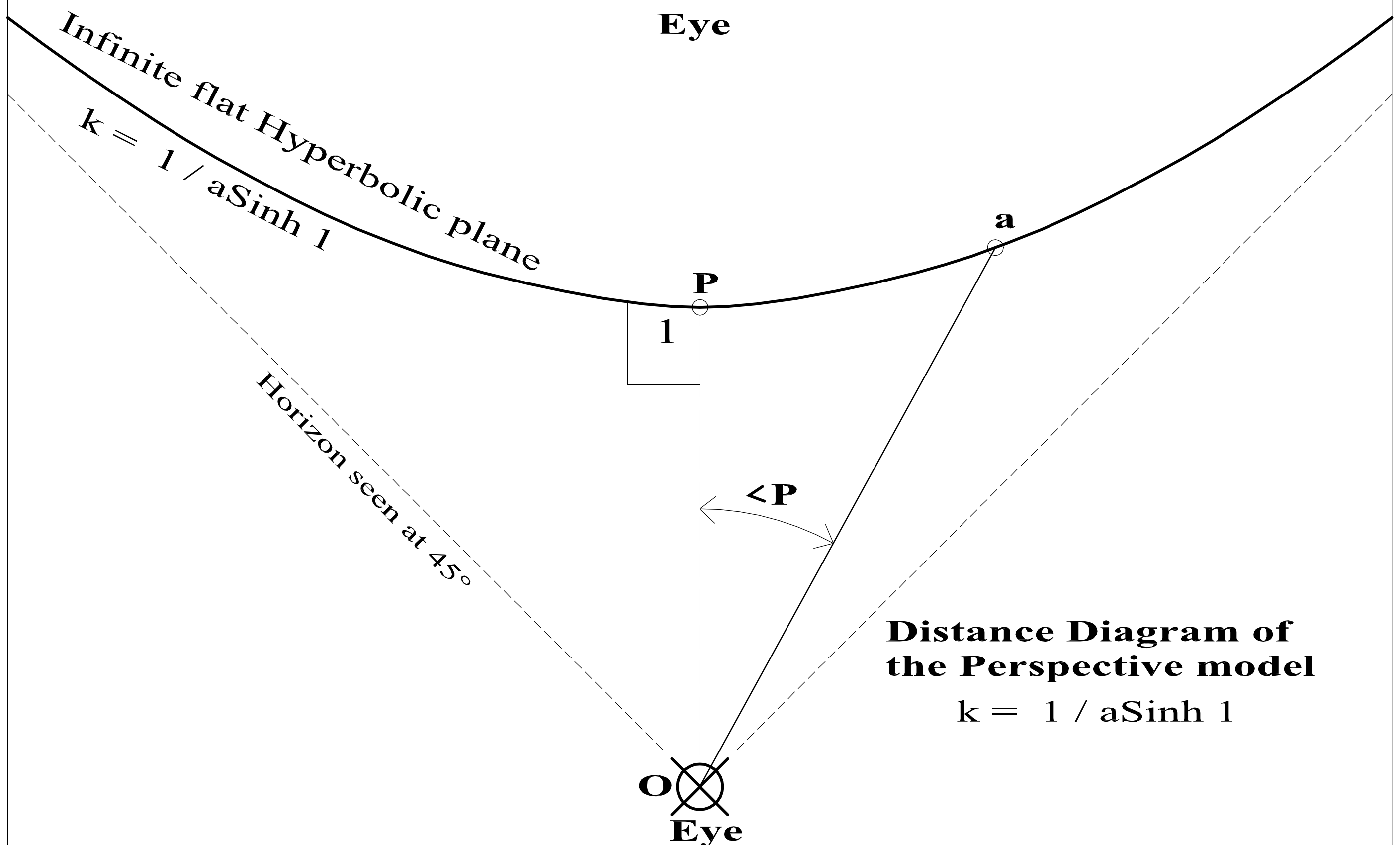
VARIATIONS OF FACTOR "k"



THE PERSPECTIVE MODEL

First of all, no standard Perspective view would be allowed for an angle as wide as the Hyperboloid's 90° spread. The geometric method of the Perspective Model works, but the resulting image is normally deemed as unacceptably visually distorted. Nevertheless ...





There is no possible arrangement where a Distance Diagram of any Perspective view is equilvalent to the Hyperboloid Model. At first glance they appear to be similar, but calculations show they simply never quite match-up to being the same curves.

THE HYPERBOLOID MODEL COMPARED TO THE PERSPECTIVE METHOD

Distance "Pa" in the Perspective model is not equal to the distance "X", "r", or arclength in the Hyperboloid Model.

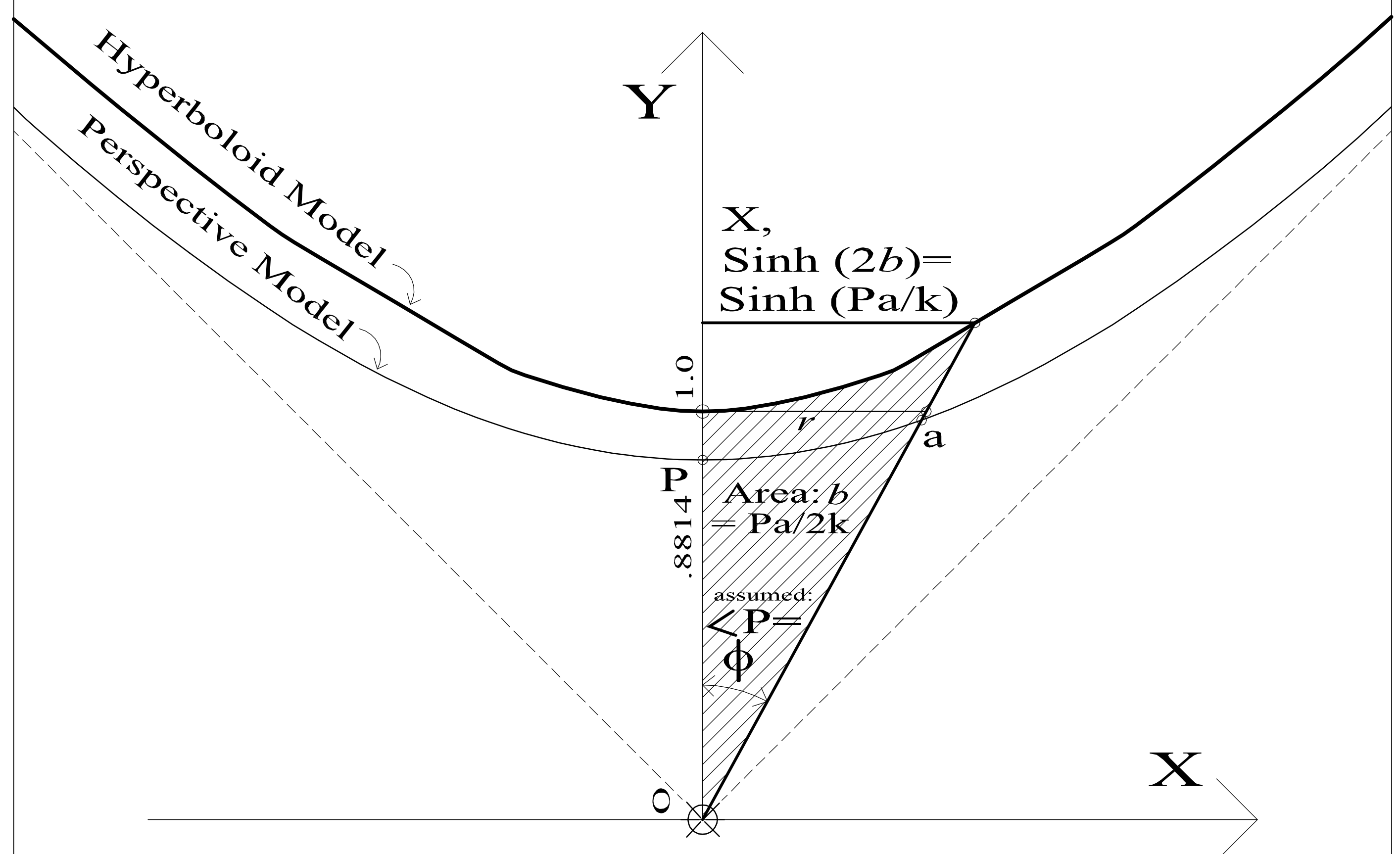
When the Perspective model is setup with the Eye at distance OP=ASinh(1) from the plane, with k=1, and with distance Pa being the distance from any given point on the flat Hyperbolic plane which is being viewed, then there is the following interesting relationships:

The Klein Model's image is exactly equivalent to the Persective image.

It is the Area "*b*" of the Hyperboloid Model that most closely relates to"Distance "Pa" of the Perspective (see below).

When the values OP, k, and Pa are all simultaneously multiplied by any factor (any constant number) then the Perspective angle <P will remain unchanged -- the Perspective image will remain unchanged -- a curious "scaling up and down" of size in the Hyperbolic Geometry where there are no "similar triangles" (such as are used to scale the size of views in Euclidean Geometry).

Another interesting relationship to note is that in this particular circumstance where OP and k are so proportioned (as stated above), the Angle of *Parallelism* for d=Pa will be $<\Pi_{Pa} = 90° - 2(<P)$.



CONCLUSION

In the wide field of "Descriptive Geometry" (the study of methods of Technical Illustration) the visualizations of the Hyperboloid Model turn out to be most closely kin to what I like to call Spherical Perspective ("Curvilinear Perspective") and global Cartography -- the difference being that in the Hyperboloid Model the sphere used for constructing the images is "turned inside out" -- its radius having a value which is an "imaginary number".

Seeing the vast array of possible illustrations developed for Cartography, it would seem safe to predict that there are possibilities for further illustration inventions that might be derived from the Hyperboloid Model, with such illustations perhaps being better suited to show geometric aspects of certain features of Hyerbolic Non-Eucldean Geometries.

I have mentioned that the image of the Klein Disk Model is matched by Perspective images constructed under certain special conditions; and it seems likely that the other visualizations already being projected from the Hyperboloid Model would also have exact duplicate images derived from other similar carefully specified Spherical Perspective projection methods. (The "Poincare Model", the "Poincare Half-Plane Model", and "Gans Model" would also -- I will guess -- have duplicate image possibilities, constructed by other spherical methods.)

AN ANALOGY:

Outside your window you notice a beautiful garden, but for some reason you are unable to get outdoors into it. You see that there is a mirrored ball in the garden, and you start doing calculations, and come to realize that you can start to understand the garden landscape layout by mathematical anaylsis of the the image reflected from the surface of the ball. Pretty cool, but why don't you just look out the window at the rest of the garden?